Exercise Sheet 4

MT454 Combinatorics

To be returned on 3rd November 2004

1. For each of the posets shown below:

   (a) Write down the general form of a matrix corresponding to an element of $I(P)$.

   (b) Verify directly (by multiplying matrices) that $I(P)$ is closed under multiplication.

   (c) Calculate the Möbius function μ of the poset by using the method just before Theorem 3.23. Check that $\mu = i^{-1}$ by computing the matrix product μi.

2. Let n be a positive integer. So $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and where $e_1, e_2, \ldots, e_r$ are positive in-

tegers. Prove that the Möbius function μ of $\mathcal{D}(n)$ is defined by

$$\mu(d,d') = \begin{cases} 1 & \text{if } d = d,' \\ (-1)^k & \text{if } d \text{ divides } d', \text{ and } d'/d \text{ is a product of} \\ & \quad k \text{ distinct primes,} \\ 0 & \text{otherwise} \end{cases}$$

for all $d, d' \in \mathcal{D}(n)$. [Hint: Sheet 3 Question 4 might be useful!]

3. Let r and s be functions from $\mathbb{N}$ to $\mathbb{R}$. Suppose that

$$r(n) = \sum_d s(d)$$

where the sum runs over the positive divisors of n. Prove that

$$s(n) = \sum_d r(n/d)\mu(d)$$

where again the sum runs over the positive divisors of n, and where

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

[Hint: Use Möbius inversion in $\mathcal{D}(n)$.]