# Exercise Sheet 8

## MT454 Combinatorics

1. Define the complete graph $K_n$ properly.

   We defined $R(s,t)$, but never wrote up a definition for the multi-coloured version. Define $R(a_1, a_2, \ldots, a_k)$.

2. 101 pigeons share 31 holes. Show that there must be a hole with at least 4 pigeons sharing it.

   State the general "pigeonhole principle". (Most books or the internet might help here).

3. Give a colouring of the $K_5$ with 2 colours so that there is no monochromatic triangle. (Try to have a picture as symmetric as possible).

   Let $G$ be a graph defined on 8 vertices. The vertices are numbered from 0 to 7. Colour the edge $(x,y)$ red if

   $$x - y \bmod 8 \in \{1, 4, 7\}.$$

   (Here $x - y \bmod 8$ is assumed to be respresented by a number from 0 to 7). The edge $(x,y)$ is coloured blue if

   $$x - y \bmod 8 \in \{0, 2, 3, 5, 6\}.$$

   Show that $G$ has no red $K_3$ and no blue $K_4$. Use this to show that $R(3,4) \geq 9$.

4. Prove that $R(s,t)$ is finite and give a concrete upper bound. Evaluate this upper bound (possibly using Stirling's formula) for $s = t$ and $s = 2t$.

5. Prove that $R(3,3) \leq 6, R(3,4) \leq 9$, and $R(4,4) \leq 18$.

6. Prove the following lower bound on the Ramsey numbers:
   $R(s,t) \geq (s-1)(t-1) + 1$.

7. Using a standard result from Ramsey theory prove: There exists an $N$ so that: Whenever $x_1, \ldots, x_N$ is a sequence of distinct integers, then the sequence contains an increasing subsequence of length 100 or a decreasing subsequence of length 100.
   Hint consider pairs $(i,j)$ with $i < j$ and $x_i > x_j$. Define a suitable colouring of $K_N$ and ...

   Which bound do you get on $N$?

   (More difficult:)
   Now forget that you know about Ramsey theory. Try to find a much better upper bound on $N$.