Exercise Sheet 9 and 10

MT454 Combinatorics

1. Study the rotations of a cube. How many are there? How many have which order? Use the orbit stabilzer theorem, applied to a) the corners, b) the faces, c) the edges.

   (Just for your information, not needed here: the rotations can also be described by studying the permutation of the 4 diagonals of the cube.)

2. Use a book or the internet to get information about the 5 platonic solids. How many faces, edges, corners have these? How many rotations are there? Which group is the group of rotations? (The latter does not need to proved, this is a rather algebraic question).

   The symmetries of some platonic solids coincide. Perhaps this simplifies your study.

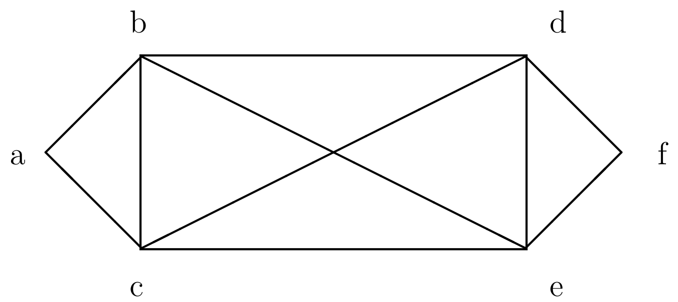
3. The *dihedral group* D_{10} *of order 10* is the set of symmetries of a regular pentagon. Suppose the corners of the pentagon are labelled $1, 2, 3, 4$ and 5 going clockwise. Write down each element of D_{10} in disjoint cycle notation, where D_{10} is regarded as a subgroup of the group S_5 of permutations of the corners. For $1 \leq n \leq 10$, how many elements of order n does D_{10} have?

4. Find all automorphisms of the graph given by the following adjacency list (drawing a nice picture of the graph will help):

1	2	3	4	5	6	7	8
2	1	1	1	2	3	4	4
3	3	2	7	7	7	5	5
4	5	6	8	8	8	6	6

   [So the first column says that 1 is adjacent to 2,3 and 4.]

5. Let V be the set of vertices of the graph Γ shown below; so $V = \{a, b, c, d, e, f\}$. Let G be the group of automorphisms of Γ, thought of as a set of permutations of V. Determine the orbits of G on V and compute the orders of G_b, G and G_a.



6. Let G be a group of permutations of a set X, let $x, y \in X$ and let $h \in G(x \to y)$. Prove that $G(x \to y) = hG_y$. Prove that if u and v are in the same orbit of G then $|G_u| = |G_v|$.

7. Use the orbit counting lemma to show that there are just 5 different necklaces that can be made from 5 white and 3 black beads. Sketch them.

8. Suppose that voting ballots consist of two holes punched in a 4×4 grid (rather than the 3×3 grid in the lectures). If the counting machine cannot distinguish between rotation and reflection, how many different types of ballot paper can be recognised by the counting machine?

9. Use the formulae of Theorems 6.20 and 6.21 to write down the cycle indices of C_{12}, D_{12} and D_{14}.

10. Let G be the group of rotational symmetries of the octahedron, regarded as a set of permutations of the corners. [An octahedron has 8 faces, 12 edges and 6 corners; see below.] Show that the cycle index of G is given by
$$\zeta_G(x_1, \ldots, x_8) = \frac{1}{24}(x_1^6 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3 + 8x_3^2).$$

11. Find the cycle index of the group of rotations of the tetrahedron, regarded as a permutation of the set of **edges**. [A tetrahedron: 4 faces, 6 edges, 4 corners. See below]

12. How many ways of colouring (a) the corners (b) the faces and (c) the edges of a tetrahedron are there, up to rotational symmetry, when 2 colours are available? [Use the cycle indices given in the lectures, as well as the last question above.]

13. How many ways, up to rotational symmetry, can the faces of a dodecahedron be coloured red and blue, so that there are 3 red and 9 blue faces?