Ex. 1

(i) Using the Handshaking Lemma prove that $m \leq \frac{n(n-1)}{2}$ for every graph, and give an example to show that equality can occur (for every n).

(ii) Show by drawing it that the graph G formed from the vertices and edges of a cube is bipartite.

(iii) (more difficult) The graph G of (ii) is isomorphic to the graph H obtained from the vector space $\mathbb{Z}_2^3$ by defining $\{u, v\}$ to be an edge in $E(H)$ if and only if u and v differ in *exactly* one coordinate. So $(0, 0, 0)$ is adjacent to $(0, 1, 0)$ and $(1, 1, 1)$ is adjacent to $(1, 1, 0)$ etc. By considering the graph formed from the vector space Z_2^4 in the same manner, show that it too is bipartite, whence so is the edge graph of a hypercube, i.e. a four-dimensional cube.

Ex. 2

(i) Define two graphs G and H with ten vertices as follows:
(a) For G draw a regular decagon (a regular 10 - sided polygon) and join the opposite vertices. So if the vertices are labelled with numbers $0, 1, 2, \ldots, 9$ clockwise, join 0 to 5, 1 to 6, etc.
(b) For H draw a pentagon with its edges (i.e. a cycle), then another smaller pentagon inside aligned in the same way (as in the Petersen Graph) and join the corresponding vertices. Each of these graphs should have $n = 10$ and $m = 15$. Are they isomorphic to each other, and is either isomorphic to the Petersen Graph?

(ii) Give an isomorphism between the two forms of the Petersen Graph.
Hint: Find a 5 - cycle in the second form as shown Figure 1.2.2 (b); see Exercises 1, Question 1 (ii).

Ex. 3

If $G = G(V, E)$ is a graph on n vertices, its complement $\overline{G} = (\overline{V}, \overline{E})$ is the graph obtained by taking $\overline{V} = V$ and defining $\{u, v\} \in \overline{E} \iff \{u, v\} \notin E$. Thus contains all the edges between the vertices of V which were not included in G.
Example: K_n is the complement of N_n.

(i) If $|E| = m$, how many edges does $\overline{G}$ have? If the valency numbers of G are $\rho_1, \ldots \rho_n$, what are the valency numbers of $\overline{G}$?

(ii) Show that the two graphs in Figure 1.4.3 are complements.

(iii) Find the one graph with 4 vertices which is isomorphic to its complement (see Exercises 1, Q4(ii)).

(iv) Find a graph with 5 vertices which is isomorphic to its complement.

Ex. 4

(i) How many non-isomorphic trees are there with (a) 6 , (b) 8 vertices?

(ii) How many of the trees in (i) (b) have $\rho \leq 4$ for every vertex v? (If the vertices denote Carbon atoms, these trees correspond to the isomers of the paraffin Octane, C_8H_{18}.)