Ex. 1
Let G be a graph with n vertices ($n \geq 2$).

(i) Show that at least two distinct vertices u, v of G have the same valency $\rho(v)$. (See Ex 1 Q3(ii)(e).)

(ii) Describe those graphs G for which every vertex v has valency $\rho(v) = 1$.

(iii) Prove that if $\rho(v) = 2$ for all the vertices v of a connected graph G, then G is isomorphic to C_n.

Ex. 2
Let G be a graph with n vertices ($n \geq 2$) and m edges.

(i) Show that for a bipartite graph G, $m \leq \frac{n^2}{4}$. Give an example with equality when n is even and find a similar upper bound and example for when n is odd.

(ii) Show that if G has an isthmus e, then it has at least two odd vertices. (Hint: Assume that G does not have any odd vertices and apply Corollary 1.3.8 to the components of $G^* = G^*(V, E \setminus \{e\})$.

(iii) Show that if the graph G is disconnected, then its complement $\overline{G}$ defined in Exercises 2 Question 3 is a connected graph. What is the diameter of $\overline{G}$ in this case?

Ex. 3

(i) Prove that if $\rho \geq 2$ for all the vertices v of a graph G, then G contains a cycle. (Notes: This can be deduced from the Handshaking Lemma 1.3.7 and Theorem 1.6.2, as used in the proof of Theorem 1.6.5), but try to find a direct proof. Although they can be proved in a similar way, this is not quite the same theorem as in Question 1(iii), though that result can easily be deduced from this one.)

(ii) Deduce from (i) that a tree with at least 2 vertices has at least one vertex v with valency $\rho(v) = 1$.

(iii) Use (ii) to give a simpler proof of Theorem 1.6.2. using weak induction on n. Deduce from this and Theorem 1.3.7 that a tree with at least 2 vertices has at least two vertices v with valency $\rho(v) = 1$.
Note: The first part of (iii) would be meaningless if (i) had been deduced from Theorems 1 and 5, as there would then be a circular argument, i.e. $P \Longrightarrow Q \Longrightarrow R \Longrightarrow P$. There are usually several ways of ordering the proofs of these theorems.

(iv) Prove that every tree with 2 vertices is a bipartite graph.

Ex. 4

(i) Let A, B, C, D be four distinct vertices. List all the labelled trees $T(V, E)$ that can be formed from $V = \{A, B, C, D\}$. For this example the edge AB is not equivalent to AC or AD etc., so that the Line Graph L_4 with vertices labelled $ABCD$ in that order is not the same as that labelled $CABD$, though it is the same as the one with vertices $DCBA$. So there are many different labelled trees which are isomorphic graphs as defined in the lectures.

(ii) How many labelled trees as defined in (i) are there for a set of five vertices?
Hint: There are far too many of them to list so try to count the numbers in each isomorphism class. For example a lot of the labelled trees will be isomorphic to the unlabelled tree L_5.