Ex. 1
By finding connected graphs with $n = 4$ or 5, show that Ore's Theorem (1.7.9) and Dirac's Theorem (1.7.10) are "best possible", i.e. that there are non-Hamiltonian graphs for which the following conditions apply.

i) $\rho(u) + \rho(v) \geq n - 1$ for every pair of non-adjacent vertices u and v.

ii) n is even and $\rho(u) \geq \frac{n-2}{2}$ for every vertex u.

iii) n is odd and $\rho(v) \geq \frac{n-1}{2}$ for every vertex v.

Note: These graphs may have Hamiltonian paths rather than cycles, i.e. be semi-Hamiltonian. Finding "good" sufficient conditions for the existence of Hamiltonian paths seems to be much harder.

Ex. 2
Euler's Formula was originally applied to the regular or Platonic solids, i.e. solid bodies for which every vertex has r edges incident with it ($\rho(v) = r \geq 3$, for all $v \in V$) and every face is a regular polygon with k edges (a k-gon).

i) Prove from the Handshaking Lemma and Euler's formula (Theorem 1.8.6) that $n(2r+2k-kr) = 4k$. Since the bracket must be positive, deduce that $(k-2)(r-2) < 4$ and hence determine the 5 possible cases. Give the values of n, m, f, r and k for each case. [Note: Two pairs of these solids are duals.]

ii) All the edge graphs of the solids in (i) are planar. For a diagram of these see http://mathworld.wolfram.com/PlatonicGraph.html

   Show that each graph is Hamiltonian.

Ex. 3

i) A solid body has s faces which are regular pentagons and t which are regular hexagons. If these are the only faces and all its vertices have valency $r = 3$, use Euler's formula to show that $s = 12$. How many hexagons are there? Do you know any examples of such bodies?

ii) Let $cr(G)$ be the crossing number of the graph G. Prove that $cr(K_5) = 1, cr(K_{3,3}) = 1$ and also prove that the Petersen graph is non-planar. (Two possible ways for the latter: count edges, vertices and use the size of the smallest cycle in connection with the technique in Theorem 1.8.8/Corollary 1.8.9), or find a "subdivision" (see page 26 of notes) of $K_{3,3}$ in the Petersen graph by deleting the vertices 0,1,2,3 in Figure 1.2.2b).)