Ex. 1

i) Check that the matrices A_1, A_2 of Example 3.2.4 satisfy the formula of Theorem 3.2.5. (i).

ii) Show that the same matrix A_1 commutes with A_1^T by finding $A_1 A_1^T$.

iii) Check thatthe parameter sets below satisfy the equations in Theorem 3.2.2. But then show, by using Theorem 3.2.5, that there are no designs with b, v, r, k, λ equal to (a) 8, 16, 3, 6, 1 or (b) 22, 22, 7, 7, 2 respectively.

iv) Find the difference set given by the quadratic residues modulo 19 . What are the parameters of the block design that it generates?

Solution:

i) Check that each column of A_1 and A_2 has three entries of 1, and that each pair of different columns have a single 1 in common in the same position, i.e. $r = 3$ and $\lambda = 1$ in both cases.

ii) Each row of A_1 has three entries of 1 and each pair of different rows have a single 1 in common in the same position, whence $A_1 A_1^T = 2I_7 + J_7 = A_1^T A_1$ from the definitions etc. Note that $r - \lambda = 2$.

iii) (a) The given numbers satisfy $\lambda v(v-1) = bk(k-1) = 1 \times 16 \times 15 = 240 = 8 \times 6 \times 5$. $vr = bk = 48$ and $\lambda(v-1) = r(k-1) = 15$, but as $1 < k < v$ and $b < v$ there is no design with these parameters from Theorem 3.2.5 (iii).
(b) Again $vr = bk = 154$ and $\lambda(v-1) = r(k-1) = 42$, but as v is even and $k - \lambda = 5$ is not a perfect square there is no design with these (symmetric) parameters from Theorem 3.2.5. (iv).

iv) The squares modulo 19 from 1 up to 9^2 are $1, 4, 9, 16, 6, 17, 11, 7$ and 5, so the difference set is $\{1, 4, 5, 6, 7, 9, 11, 16, 17\}$ with design parameters $(b, v, r, k, \lambda) = (19, 19, 9, 9, 4)$, respectively. Check: Note that 1 occurs as a difference four times: $1 = 5 - 4 = 6 - 5 = 7 - 6 = 17 - 16$, $3 = 4 - 1 = 7 - 4 = 9 - 6 = 1 - 17$ mod 3 etc.