Exercise Sheet 2

Topics in Discrete Mathematics/Number theory

2016

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Ex. 1
Let A and B be non-empty sets of integers. Suppose that $|A+B| = |A| + |B| - 1$. Prove that both A and B are arithmetic progressions with the same distance.

Ex. 2 (Nathanson)
The subset sum of a finite set A' of integers is defined by $s(A') = \sum_{a \in A'} a$. For any finite set A of positive integers, define $S(A) = \{s(A') \mid A' \subseteq A, A' \neq \emptyset\}$. Prove that if A is a set of k positive integers, then

$$|S(A)| \geq \binom{k+1}{2}.$$

Let A be a set of $k \geq 3$ positive integers such that $|S(A)| = \binom{k+1}{2}$. Prove that there exists a positive integer m such that

$$A = \{m, 2m, 3m, .., km\}.$$

Does this also hold for $k = 2$?

Ex. 3 (Nathanson)
Let A be a finite subset of the abelian group G, and let B be a finite subset of the abelian group H. The map $\phi : A \to B$ is a Freiman isomorphism if ϕ is a one-to-one correspondence between A and B and if the map $\Phi : 2A \to 2B$ defined by

$$\Phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)\}$$

is well-defined and a one-to-one correspondence. Fix $r > 5$, and let

$$A = \{0, 1, 2, r, r+1, 2r\} \subset \mathbb{Z}.$$

Show that $|2A| = 3|A| - 3 = 15$. Let

$$B = \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,0)\} \subseteq \mathbb{Z}^2.$$

Show that $|2B| = 3|B| - 3 = 15$. Construct a Freiman isomorphism between A and B.

Ex. 4
Read the paper "Adding distinct congruence classes modulo a prime" by Alon, Nathanson and Ruzsa. American Mathematical Monthly Volume 102, Number 3, March, 1995, pages 250-255 (available on "jstor" (logged in to TU with vpn) or Noga Alon's webpage).

Hand in solutions to problems 1-3 this coming Monday.

If you find typos or errors on the problem sheet please send me an email.