Exercise Sheet 3

Topics in Discrete Mathematics/Number theory

2016

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Ex. 1
Go through the details of the Lemma in Christian Reiher's paper on Kemnitz' conjecture
Lemma. If $|X| = 4p - 3$ and $(p|X) = 0$, then $(p-1|X) \equiv (3p1|X) \bmod p$.

Ex. 2
Read Thm 6.3 (and its proof) of the paper "Combinatorial Nullstellensatz" by Noga Alon (Combinatorics, Probability and Computing, 8, January 1999, 7-29, available on the Cambridge university press webpage (logged into TU with vpn) or Noga Alon's webpage).
Theorem 6.3. Let $H_1, H_2, \ldots, H_m$ be a family of hyperplanes in $\mathbb{R}^n$ that cover all vertices of the unit cube $\{0,1\}^n$ but one. Then $m \geq n$.

Ex. 3
Let

$$T(n) = T = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\2\end{pmatrix} \begin{pmatrix}1\\1\\2\\0\end{pmatrix} \begin{pmatrix}1\\1\\2\\2\end{pmatrix} \begin{pmatrix}1\\2\\0\\1\end{pmatrix} \begin{pmatrix}1\\2\\1\\0\end{pmatrix} \begin{pmatrix}1\\2\\1\\2\end{pmatrix} \begin{pmatrix}1\\2\\2\\1\end{pmatrix} \begin{pmatrix}1\\3\\1\\1\end{pmatrix}$$
$$\begin{pmatrix}2\\2\\0\\0\end{pmatrix} \begin{pmatrix}2\\2\\0\\2\end{pmatrix} \begin{pmatrix}2\\2\\2\\0\end{pmatrix} \begin{pmatrix}2\\2\\2\\2\end{pmatrix} \begin{pmatrix}2\\1\\0\\1\end{pmatrix} \begin{pmatrix}2\\1\\1\\0\end{pmatrix} \begin{pmatrix}2\\1\\1\\2\end{pmatrix} \begin{pmatrix}2\\1\\2\\1\end{pmatrix} \begin{pmatrix}2\\0\\1\\1\end{pmatrix} \begin{pmatrix}0\\2\\1\\1\end{pmatrix} \begin{pmatrix}3\\1\\1\\1\end{pmatrix}$$

Write a computer programme and verify, that modulo an odd integer $n = 3, 5, 7, \ldots$ the 20 vectors in $\mathbb{Z}_n^4$, each taken $n-1$ times do not have a zero sum of length n, for odd n. (See how far you can go with your programme, with regard to increasing n). The proof technique of the three dimensional case also works here, but is more tedious).
On the other hand, see which vectors $v \in \mathbb{Z}_n^d$ you can reach with a sumset of $1, 2, \ldots, n$ summands (again, for fixed odd $n = 3, 5, \ldots$).

(E.g., fixing $n = 5$, start with 20 vectors, how many distinct vectors do you get as a sum of any 2, 3, 4, 5 vectors? For 5 recall that there are no 5 copies of any fixed vector.)
Observing the first coordinate, 9 times 1, 9 times 2 and then 0 and 3, one can prove that any zero sum of length n would require the last two vectors.

Ex. 4
If one allows zero sums of length kp (rather than p only), one can say more. There are for example results by Silke Kubertin, again using the polynomial method.

Hand in solutions to problem 3 on Monday 9th May.

If you find typos or errors on the problem sheet please send me an email.