Exercise Sheet 4

Topics in Discrete Mathematics/Number theory

2016

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Ex. 1

This refers to the paper that we discussed in class: “Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small” (Ernie Croot, Vsevolod Lev, Peter Pach) http://front.math.ucdavis.edu/1605.01506

a) Let $n = 4, d = 3, P(x_1, x_2, x_3, x_4) = 2x_1x_2 + 3x_1x_2x_3 + x_4$.
Determine $K, P((x_1, x_3, x_4) - (y_1, y_2, y_3, y_4))$ (also with the split into two parts, as in the paper) $\vec{u}$ and $\vec{v}$.

b) Look up, where the inequality $\sum_{0\le i\le z} \binom{n}{i} < 2^{H(z/n)}$, where H is the entropy function defined by $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ for $x \in (0, 1)$, comes from.

(Closely related are “large deviation” inequalities, estimating the tail of a rapidly decaying function, such as the boundary of a binomial distribution).

c) Verify the proof of Theorem 1 (we did not check all details from Calculus).

Ex. 2

For small dimension work out the maximal size $|A|$ of a set set $A \subset \mathbb{Z}_4^n$ without a proper 3-progression. (For the maximum size $|A|$ are there several non-isomorphic examples?)

For large dimension, can you give a general construction, satisfying $|A| \ge 2^n, |A| \ge 2.5^n, |A| \ge 2.99^n, |A| \ge 3.2^n$ etc?

Ex. 3

Read the closely related article “On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression” by Jordan S. Ellenberg, Dion Gijswijt.
http://front.math.ucdavis.edu/1605.09223
and also comments on the blogs such as https://gilkalai.wordpress.com/2016/05/15/mind-boggling-following-the-work-of-croot-lev-and-pach-jordan-ellenberg-settled-the-cap-set-problem/
https://quomodocumque.wordpress.com/

https://gowers.wordpress.com/2016/05/19/reflections-on-the-recent-solution-of-the-cap-set-problem-i/
https://terrytao.wordpress.com/2016/05/18/a-symmetric-formulation-of-the-croot-lev-pach-ellenberg-gijswijt-capset-bound/

Hand in solutions to problems 1,2 on Monday 6th June.

If you find typos or errors on the problem sheet please send me an email.