Exercise Sheet 1 COMBINATORICS
2014/15

Ex. 1
Let $\mathrm{ex}(n, C_3)$ denote the maximum number of edges a graph on n vertices can have, without containing a triangle (=cyclic graph of length 3) as a subgraph. Determine $\mathrm{ex}(n, C_3)$ as good as possible.

Ex. 2
Let $\mathrm{ex}(n, C_4)$ denote the maximum number of edges a graph on n vertices can have, without containing a cyclic graph of length 4.
Let p be prime, and let the set of vertices V be $V = \mathbb{Z}_p \times Z_p$. The vertices (x, y) and (x', y') are joined by an edge if $x + x' = yy'$. Show that (x, y) has at least $p - 1$ neighbours. From this, show that $|E| \geq \frac{1}{2}p^2(p-1)$. Show that there is no 4-cycle. (Suppose that (x, y) has two distinct neighbours $(x_1, y_1) \neq (x_2, y_2)$, (say). Show that, by the definition of the edges, y and then also x is uniquely defined.)
Would the same type of proof work if p is not a prime? (if yes, for which other values does it work?, if no, why not?)
Compare the new result on $\mathrm{ex}(n, C_4)$ with the previous ones.

Ex. 3
Given a set of n distinct points $p_1, \ldots p_n \in P$ in the Euclidean plane, and a set of n distinct lines $l_1, \ldots, l_n$. Prove that the number of point-line incidences (p_i, l_j) with $p_i \in l_j$ is $O(n^{3/2})$.
Hint: use the graph theoretic results we studied so far.

Ex. 4
Verify the number theoretic details (Lemma 1.8) of the proof of the Bruck-Ryser-Chowla theorem, see e.g. lecture notes by Simeon Ball, Zsuzsa Weiner, linked to on webpage.

Ex. 5 (Maybe with computer)
Give a list of 4 mutually orthogonal Latin squares of order 5.

Now let $p = 3$. Find an irreducible quadratic polynomial over $\mathbb{F}_3$. From this construct a finite field of 9 elements. (give its addition and multiplication table). From this, maybe with computer, give a list of 8 mutually orthogonal latin squares of order 9.

Ex. 6 (With computer)
Determine the number of Latin squares of order 6, which are essentially distinct. (Think about "normalizing" the Latin squares.) Try to prove: There are no two orthogonal Latin squares of order 6.

[Maybe you should start counting problems which are way too hard and tell me later the result...]
Please cross the problems you did in the Kreuze-system, link at webpage below. Deadline, Tuesday 21st October, 10am.

http://www.math.tugraz.at/~elsholtz/WWW/lectures/ws14/kombinatorik/vorlesung.html

Please note that by Nawi-regulatations, everybody who takes part in the exercise programme will receive a grade. (Deregistration from the Übungen is possible during October).

Exercise Sheet 2 COMBINATORICS
2014/15

Ex. 7
Study the affine and the projective plane of order 4, i.e. give a list of all lines, and try to visualize them.
(Hint: you have to work with the finite field $\mathbb{F}_4$, not with $\mathbb{Z}_4$).

Ex. 8
A projective space satisfies the following axioms:

P1) For any pair of two points, there is a unique line going through it.

P2) Let p, q, r, s be four distinct points. If the lines pq and rs intrsect, then the lines pr and qs intersect as well.

P3) Each line contains at least three points.

P4) There exist three points, not all on one line.

Further, let V be a vector space of dimension $d \geq 3$ over a field of q elements. Let $U_1(V)$ and $U_2(V)$ denote the one- and two-dimensional subspaces of V. Let $P(V) = (U_1(V), U_2(V), \subseteq)$ denote a geometry, where the points are the one-dimensional subspaces, the lines are the two-dimensional subspaces, with canonical subset-inclusion.
a) Discuss the difference to a projective plane.
b) Prove that $P(V)$ is a projective space.
c) Prove that $P(V)$ is a $2 - (q^{d-1} + q^{d-2} + \cdots + q + 1, q + 1, 1)$ design.

Ex. 9
Study a three-dimensional projective space over $\mathbb{F}_2$, i.e. determine the number of points, lines, and describe the lines. (How do the planes look like? How many are there?)

Ex. 10
Prove the existence of a Hadamard-matrix of order 2^n by verifying the construction below: Let $X = \{1, \ldots, n\}$. Let $S_1, \ldots, S_{2^n}$ be the 2^n subsets of X, (in any order). Define $H = (h_{ij})$ by:

$$h_{ij} = (-1)^{|S_i \cap S_j|}.$$

Ex. 11
a) Prove: if Hadamard matrices of order n_1 and n_2 exist, then a Hadamard matrix of order $n_1 n_2$ exists.

b) Find a Hadamard matrix of order 12.

Ex. 12 (Small excursion to coding theory)
Let C be the set of words of length 8, consisting of all cyclic shifts of the three words below $11010000, 11100100, 10101010$, and of $00000000, 11111111$. (What

is $|C|$?).
a) Show that any two words differ in at least two positions. How many comparisons are required to do this? (The real answer is smaller than the trivial upper bound).
b) Try the following:
b1) either find a set with more than $|C|$ words, where any two elements differ in at least three positions.
b2) or prove that the number $|C|$ is the maximum number with this property.