Exercise Sheet 5 COMBINATORICS
2014/15

Ex. 28

a) Prove that $R(3,4) \leq 10$, and with an extra thought: $R(3,4) \leq 9$. Hint: as started in class: Start with 10 vertices. Any vertex V has 9 neighbours. suppose V has 6 (or more) neighbours connected along red edges. Then by $R(3,3) = 6$ there must be either a red K_4 or a blue K_3. Hence V has at most 5 red adjacent edges. Suppose V has at least 4 neighbours, connected along blue edges, then again either there is a red K_4 or a blue K_3.

This proves $R(3,4) \leq 10$.

Now, analyzing the proof above, and starting with 9 vertices. What can you say about the colour of the adjacent edges? In particular prove that it is not possible that all vertices have exactly five adjacent red edges. Hence conclude that $R(3,4) \leq 9$.

c) Let G be a graph defined on 8 vertices. The vertices are numbered from 0 to 7. Colour the edge (x, y) blue if

$$x - y \bmod 8 \in \{1, 4, 7\}.$$

(Here $x - y \bmod 8$ is assumed to be respresented by a number from 0 to 7). The edge (x, y) is coloured red if

$$x - y \bmod 8 \in \{0, 2, 3, 5, 6\}.$$

Show that G has no red K_4 and no blue K_3. Use this to show that $R(3,4) \geq 9$.

Ex. 29

Now taking the idea of proof of the bound $R(3,4) \leq 10$ above, and prove that $R(s,t) \leq R(s-1,t) + R(s,t-1)$. Comparing with the binomial coefficients evaluate a concrete upper bound (for example using Stirling's formula) for $s = t$ and $s = 2t$, as t goes to infinity.

Ex. 30

a) Prove that $R(4,4) \leq 18$, (easy).

b) Let $X = \{0, 1, 2, ..., 16\}$ be the set of residues mod 17 and let G be the complete graph on X. Given $x, y \in X$ with $x < y$, colour the edge $\{x, y\}$ red if $y - x$ is equal to a square number modulo 17, and blue otherwise. (For example, $\{2, 10\}$ is red because $10 - 2$ mod 5^2 mod 17.)
b1) Find all square numbers modulo 17.
b2) Show that if $x, y, u \in X$ then $\{x, y\}, \{x + u, y + u\}$ and $\{xu^2, yu^2\}$ all have the same colour. (Note that one can assume that $u \neq 0$).
b3) Prove that G has no monochromatic 4-set. [Hint: use (b2) to reduce the number of cases that have to be considered.]
b4) What does this imply about $R(4,4)$?

Ex. 31

a) Show that an arbitrary 2-colouring of an 4×7 board contains the four corners of a a monochromatic (axis-parallel) rectangle. What about colouring of an 4×6 board?

b) Show that there exist m and n such that: an arbitrary 2-colouring of an $m \times n$ board contains the 9 squares of a monochromatic (axe-parallel) 3×3 subgrid.
For example, columns 1,2,5 in lines 1,3,4 define such a grid).

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XXOOX
OOXXX
XXXXX
XXXOX
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For information:
In his book "Timaios" Plato proves that there are at most 5 regular solids. The proof is a simple folding technique from plane to three-space. He does not discuss why (for example) a solid like the icosahedron, folded just on one corner, actually exists.
`http://12koerbe.de/pan/timaios6.htm` Timaois auf Deutsch
and
`http://www.math.tugraz.at/~elsholtz/WWW/lectures/ss00/kogeo/platon1.pdf`
For information
Book XIII of "The Elements" of Euclid shows the existence of the Platonic solids, by constructions. For example, the lengths of the sides of the icosahedron and dodecahedron are calculated.
`http://aleph0.clarku.edu/~djoyce/java/elements/bookXIII/bookXIII.html`
The classical translation with commentary is by Thomas Heath (based on the best available source, a comparison of various handwritten manuscripts, which are actually slightly different versions of the text, due to Heiberg)
`http://www.wilbourhall.org/pdfs/Heath_Euclid_III.pdf` (large file!), starting at page 481.

Ex. 32
A much simpler construction of the icosahedron is due to Luca Pacioli (14451517): Look it up: (pages 22 and 24) of John Stillwell, Mathematics and Its History

`http://books.google.at/books?id=3bE_AAAAQBAJ&printsec=frontcover&hl=de#v=onepage&q&f=false`
or `http://en.wikipedia.org/wiki/Golden_rectangle`
The construction starts with three (orthogonally placed) "golden" rectangles, of side ratio $\frac{\sqrt{5}+1}{2} : 1$. Prove that the resulting triangles (see triangle ABC in Stillwell's book) are equilateral, and then observe by symmetry that all triangles are of the same type.
Let $\tau = \frac{\sqrt{5}+1}{2} = 1.61\ldots$ The twelve vertices of the icosahedron (with edge length 2) are:

$$(0, \pm\tau, \pm 1), (\pm 1, 0, \pm\tau), (\pm\tau, \pm 1, 0).$$

The twenty midpoints of the faces (and therefore, also the corners of a dodecahedron are:

$$\frac{\tau^2}{3}(0, \pm\frac{1}{\tau}, \pm\tau), \frac{\tau^2}{3}(\pm\tau, 0, \pm\frac{1}{\tau}), \frac{\tau^2}{3}(\pm\frac{1}{\tau}, \pm\tau, 0), \frac{\tau^2}{3}(\pm 1, \pm 1, \pm 1)$$

The thirty midpoints of edges are:

$$(\pm\tau,0,0),(0,\pm\tau,0),(0,0,\pm\tau),\frac{1}{2}(\pm\tau^2,\pm1,\pm\tau),\frac{1}{2}(\pm\tau,\pm\tau^2,\pm1),\frac{1}{2}(\pm1,\pm\tau,\pm\tau^2).$$

In principle, one can work out the 60 rotation matrices. All you need to recall is a formula of this matrix, given the axis, and given the rotation angle:
If (n_1,n_2,n_3) with $n_1^2+n_2^2+n_3^2=1$ is the rotation axis:

$$\begin{pmatrix} \cos\theta+n_1^2(1-\cos\theta) & n_1n_2(1-\cos\theta)-n_3\sin\theta & n_1n_3(1-\cos\theta)+n_2\sin\theta \\ n_1n_2(1-\cos\theta)+n_3\sin\theta & \cos\theta+n_2^2(1-\cos\theta) & n_2n_3(1-\cos\theta)-n_1\sin\theta \\ n_1n_3(1-\cos\theta)-n_2\sin\theta & n_2n_3(1-\cos\theta)+n_1\sin\theta & \cos\theta+n_3^2(1-\cos\theta) \end{pmatrix}.$$

One more thing: from Pacioli's construction one can easily see that one can circumscribe a cube around the icosahedron, just use the 6 sides of the "golden" rectangles. For a picture see here:
`http://www.georgehart.com/virtual-polyhedra/ex-pr1.html`

Similarly one can inscribe cubes, and the rotations correspond to the even permutations of cubes inside (or outside) in five distinct positions, (giving the 60 even permutations of A_5).
Information: The classification of the finite subgroups of the group of rotations SO_3 is due to Felix Klein. You can view at his famous book "Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade" here: `https://archive.org/stream/vorlesungenber00kleiuoft#page/n5/mode/2up`
starting on page 16 the group of rotations of the icosahedron is studied, and the isomorphy to the group A_5 is shown.
(There is much information above, for the purpose of the exercise in class: Just do (at least) Pacioli's construction, and and write down 3 corresponding rotation matrices, one of order 2,3, and 5.) If you have time try to understand why there are cubes inside the icosahedron/dodecahedron (maybe also search for pictures on the internet).

Ex. 33
A_5 is smallest nonabelian "simple" group. A group is simple if it has only trivial "normal subgroups".
Prove that A_5 is simple by studying the even permutations. (A permutation is even, if the number of transpositions describing the permutation is even. $(123)=(13)(12)$ is even.) List all conjugacy classes, (they are related to

those of S_5, even though in some cases a class in S_n splits into two classes in A_n). Note that a normal subgroup must contain the union of (complete!) conjugacy classes, and that there is no combination of these which could possibly be a subgroup of A_5, according to Lagrange's theorem.

An alternative, geometric proof!
A Simple Proof for the Simplicity of A_5 Benno Artmann, American Math. Monthly 95, no 4. (1988) 344–349.
`http://www.jstor.org/stable/2323573` (For this link you need to be logged into TU/library account).
The author proves that all rotations of order 2 are conjugate, and generate A_5. The same for rotations of order 3. For order 5, rotations by $2\pi/5$ and $4\pi/5$ are not conjugate, but they all generate A_5.