Exercise Sheet 6 COMBINATORICS
2013/14

Ex. 34
Using a standard result from Ramsey theory prove: There exists an N so that: Whenever $x_1, \ldots, x_N$ is a sequence of distinct integers, then the sequence contains an increasing subsequence of length 100 or a decreasing subsequence of length 100.
Hint: consider pairs (i, j) with $i < j$ and $x_i > x_j$. Define a suitable colouring of K_N and ...

Ex. 35
(Using a standard result from Ramsey theory prove: There exists an N so that: Whenever $x_1, \ldots, x_N$ is a sequence of distinct integers, then the sequence contains an increasing subsequence of length 100 or a decreasing subsequence of length 100.
Hint: consider pairs (i, j) with $i < j$ and $x_i > x_j$. Define a suitable colouring of K_N and ...)
Now prove a best possible result on this question: Whenever $x_1, \ldots, x_N$ is a sequence of $N = n^2 + 1$ distinct integers, then the sequence contains an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n + 1$.
Hint: these numbers look like applying a pigeonhole principle with an $n \times n$ square. One way to do this is to prove that the following algorithm works.
Place the N elements in columns and in these columns on top of each other as follows:
1) x_1 is in the first column.
Now for $i = 2, \ldots, N$:
2) if x_i is larger than the top value of an already used column, place x_i on top of the first such column.
3) Otherwise start a new column.
Also, study the following sequence with this algorithm: 7,1,9,14,5,8,16,11,4,2,12,15,6,13,17,3,10

Ex. 36
One can get multicolour Ramsey numbers such as $R(k, k, k)$, with three colours, by combining two colours to one colour and iterating a Ramsey argumnet on $R(k, l)$ with $l = R(k, k)$. Which upper bound does one get on $R(k, k, k)$? Can you do better than this?
Try to use the probabilistic method to get a lower bound on $R(k, k, k)$

Ex. 37
Let the integers of the interval $[1,n]$ be coloured red and blue, assume that n is large). Show that there *many* monochromatic Schur-triples, ($\geq cn^2$). What is the worst case colouring, i.e. for which colouring is c as small as possible, i.e. for all colourings the number of Schur-triples is $S(n) \geq (c+o(1))n^2$. (Maybe you cannot fully prove that it's the worst case).

Ex. 38
According to van der Waerden's theorem, for any positive integers r and k there exists a positive integer N such that if the integers $\{1,2,...,N\}$ are r-coloured, (an r-colouring is a map $\chi : \{1,\ldots,N\} \to \{1,\ldots,r\}$), then there is a monochromatic arithmetic progression of length (at least) k. The van der Waerden number $W(r,k)$ is the least integer N with this property.
Determine the least integer $N = W(2,4)$ such that every 2-colouring of $[1,N]$ contains a monochromatic arithmetic progression of length 4.

Update: you can find patterns of length 34 without 4-progressions, on the internet, or by computation. If you are good with programming try to prove that for length 35 there is always a 4-progression. Otherwise give an upper bound for length of intervals without 4-progressions, (even if this bound is quite weak), for example by following the general proof strategy of van der Waerden. Here you can use (without further proof) that $W(r,3)$ is finite. (see also the next exercise).

Ex. 39
Give an explicit upper bound for $W(3,3)$, (following the proof in class). Try to improve it.

Ex. 40
Let $S_{l,k} \subset [1,N]$ with $N = 5^k$ be the set of integers using in a base 5 presentation only digits $0,1,2$, and using exactly l "ones". Show that the set $S_{l,k}$ does not contain three integers in arithmetic progression. For very large $N = 5^k$ optimize the parameter l to give a set which contains more integers than were used by Erdős-Turán (or Szekeres) (who used in base three representations of integers with digits 0 and 1).

Ex. 41
Let $S_{d,k,r} \subset [1,N]$ with $N = (2d-1)^k$ be the set of integers using in a base $2d-1$ presentation only digits $0,1,2,\ldots,d-1$, lying on a sphere:

$$S_{d,k,r} = \{n = \sum_{i=0}^{k} a_i(2d-1)^i : 0 \leq a_i \leq d-1 \text{ and } \sum_i a_i^2 = r.\}$$

Show that the set $S_{d,k,r}$ does not contain three integers in arithmetic progression. For very large N optimize the parameter d, k, r to give a set which contains as many integers as you can find in this way. (Hint: you can try $d \sim c(\log N)^{\alpha}$.)

Ex. 42
Let N be a huge integer, and $\alpha \in (0,1)$ be a real constant. Let $S \subset [1,N]$ with $|S| \geq \alpha N$. Prove that there is a Hilbert cube $H(a_0; a_1, \ldots a_d) \subset S$ with $d \geq c_\alpha \log\log N$, (even $d \geq \log\log N + O_\alpha(1)$ might work).

(Hint: forget about the colours of the lecture notes. Rather try to understand how one can choose good values a_1, a_2 etc. Try: a_1 is the most frequent difference between any two elements in S. Now, choose a_2 in a similar "greedy" way. How many copies of a small cube with a_1, a_2 does one have? How can one continue this?)