Number theory exercises WS 2019, TU Graz

Sheet 2, solutions (on paper) to be handed in on 22th October 2019

2-1. Prove: If $a^n + 1$ is prime, then $n = 2^k$, where $k \geq 0$ is an integer. (Hint: compare with: if $2^n - 1$ is a prime, then n is a prime.)

2-2. Prove that

$$\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1.$$

2-3. If $p \mid 2^{2^n} + 1$, then $p \equiv 1 \bmod 2^{n+1}$. Conclude that for all $n \geq 2$ there exist infinitely many primes $p \equiv 1 \bmod 2^n$,

2-4. Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a so called Dirichlet series. Here $a_n \in \mathbb{C}$ are coefficients and s is a complex variable. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. The series is convergent when $\mathrm{Re}(s) > 1$. Prove that

$$(\zeta(s))^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}.$$

2-5. Prove that $\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$. (Counting of lattice points.) You can use for example Apostol, Introduction to analytic number theory, Theorem 3.3.
(NOT for crosses, and not to be handed in.)

Deadline for crosses are: Tuesday 9.55am.

https://www.math.tugraz.at/~elsholtz/WWW/lectures/ws19/numbertheory/vorlesung.html