Number theory exercises WS 2019, TU Graz

Sheet 4, solutions (on paper) to be handed in on 19th November 2019

4-1. Find the Babylonian clay tables that list Pythagorean triples (Plimpton 322).

4-2. Find (online or in library) various proofs of a Theorem of Fermat that there do not exist four squares in arithmetic progression, i.e. not all four values $a, a+d, a+2d, a+3d$ can be squares. Then look at these proofs until you find one which is easy enough to understand with your mathematical knowledge.

a) Give a list of all proofs you find, and write some key words such as elementary, uses elliptic curves, etc.

b) Work through the proof that you understand in detail. so that you can present it on the board (maybe as a team of 2).

4-3. Consider the equation $x^2 + 3y^2 = z^2$ and primitive solutions (x, y, z), (i.e. $\gcd(x, y, z) = 1$).

a) Show that x has to be odd, (Hint; work modulo 4)

b) Show: if y is odd, then z is even.

If $3 \mid z - x$, write $y^2 = \frac{z-x}{3}(z + x)$.

Conclude that both factors must be integer squares.

Hence there exist integers r, s such that

$$x = \frac{r^2 - 3s^2}{2}, y = rs, z = \frac{r^2 + 3s^2}{2}.$$

c) If y is evenm, there exist integers m, n such that $x = m^2 - 3n^2, y = 2mn, z = m^2 + 3n^2$.

4-4. Show there are no solutions to $x^3 + y^3 = 3z^3$ in positive integers.

Hand in solutions to problems 4-2, 4-3, 4-4

Deadline for crosses are: Tuesday 9.55am.

`https://www.math.tugraz.at/~elsholtz/WWW/lectures/ws19/numbertheory/vorlesung.html`