COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 24 (3) (2004) 351–358

LOWER BOUNDS FOR MULTIDIMENSIONAL ZERO SUMS

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Received March 13, 2001

Let $f(n,d)$ denote the least integer such that any choice of $f(n,d)$ elements in $\mathbb{Z}_n^d$ contains a subset of size n whose sum is zero. Harborth proved that $(n-1)2^d+1 \le f(n,d) \le (n-1)n^d+1$. The upper bound was improved by Alon and Dubiner to $c_d n$. It is known that $f(n,1)=2n-1$ and Reiher proved that $f(n,2)=4n-3$. Only for $n=3$ it was known that $f(n,d)>(n-1)2^d+1$, so that it seemed possible that for a fixed dimension, but a sufficiently large prime p, the lower bound might determine the true value of $f(p,d)$. In this note we show that this is not the case. In fact, for all odd $n\ge 3$ and $d\ge 3$ we show that $f(n,d)\ge 1.125^{\lfloor \frac{d}{3}\rfloor}(n-1)2^d+1$.

1. Introduction

A classical result of Erdős, Ginzburg, and Ziv [8] states that amongst any $2n-1$ integers one can choose n such that their sum is divisible by n.

Harborth [14] considered the corresponding problem for d-dimensional integer lattices. Let $f(n,d)$ denote the minimal number such that any choice of $f(n,d)$ not necessarily distinct vectors $v_i \in \mathbb{Z}_n^d$ contains a subset of n vectors whose sum is $0\in\mathbb{Z}_n^d$. Harborth proved that

$$(n-1)2^d+1 \le f(n,d) \le (n-1)n^d+1.$$

The lower bound follows from the example in which there are $n-1$ copies of each of the 2^d vectors with entries 0 or 1. The upper bound follows since any set of $(n-1)n^d+1$ vectors must contain, by the pigeonhole principle, n vectors which are equivalent modulo n.

Mathematics Subject Classification (2000): 11B50; 05D05

The upper bound was greatly improved upon by Alon and Dubiner [2]:

$$f(n,d) \le c_d n.$$

For composite integers $n = n_1 n_2$ an upper bound on $f(n,d)$ can be derived by bounds on $f(n_1,d)$ and $f(n_2,d)$ as follows (see Harborth [14]):

$$(1)\quad f(n_1 n_2, d) \le \min\{f(n_1,d)+(f(n_2,d)-1)n_1, f(n_2,d)+(f(n_1,d)-1)n_2\}.$$

One therefore often restricts the consideration to a prime argument $n = p$.

For $d=2$ it was conjectured that $f(n,2)=4n-3$, so that the lower bound would determine the correct value. In fact it suffices to prove this conjecture for prime values of n, since the conjecture then follows by (1). There were partial results in favour of this conjecture, see Kemnitz [16], as well as Gao [10], [11], [12], and Thangadurai [21].

Rónyai [20] proved that for primes p one has $f(p,2) \le 4p-2$, which implies that $f(n,2) \le 4.1n$. Gao [13] extended this to powers of primes: $f(p^a,2) \le 4p^a - 2$. Reiher [19] eventually proved this conjecture: $f(n,2)=4n-3$.

For $d \ge 3$ very little is known.

$f(2^a,d) = (2^a-1)2^d+1$	(see Harborth [14]),
$f(3,3) = 19$	(see Harborth [14], Brenner [3]),
$f(3,4) = 41$	(see Pellegrino [18], Brown and Buhler [4], Brenner [3], Kemnitz [16]),
$91 \le f(3,5) \le 121$	(see Kemnitz [15]),
$f(3,5) = 91$	(see Edel et al. [7], [5])
$225 \le f(3,6) \le 229$	(see Edel et al. [7], [5])
$f(3,18) \ge 300 \times 2^{12}$	(see Frankl, Graham, Rödl [9]),
$f(3,d) \ge 2.179^d$ for $d \ge d'$	(see Frankl, Graham, Rödl [9]),
$f(3,d) \ge 2.217389^d$ for $d \ge d'$	(see Edel [6])
$f(n,d) = o(n^d)$ for fixed n, as $d \to \infty$	(see Alon, Dubiner [2])

where d' is sufficiently large. The result $f(3,d)=o(3^d)$ (Brown and Buhler [4], and Frankl, Graham and Rödl [9]) can be derived from the Szemerédi–Roth theorem on arithmetic progressions of length 3. Ruzsa proved that $f(3,d) = O\left(\frac{3^d}{\sqrt{d}}\right)$ holds (see Meshulam [17]) and Meshulam proved a more general result on arithmetic progressions of length three in finite abelian groups which implies as a special case $f(3,d) \le 2\frac{3^d}{d}$ (see also [17]).

The exact determination of $f(n,d)$ is a very difficult problem. Harborth's lower bound has been improved only in the above mentioned very few special cases with $n=3$. Not a single case for odd $n>3$ was known with $f(n,d) > 2^d(n-1)+1$. It seemed conceivable that for any fixed dimension and a sufficiently large prime p the lower bound determines the correct value of $f(p,d)$. In this note we show that this is not the case. We prove the following theorem:

Theorem. *Let $n \geq 3$ be an odd integer. The following inequality holds:*

$$f(n,d) \geq 1.125^{\lfloor \frac{d}{3} \rfloor}(n-1)2^d + 1.$$

Formally, the bound is also valid for $d=1$ and $d=2$, but is not new in these cases. This theorem has a number of simple corollaries:

Corollary 1. *For fixed d this implies that the constant c_d in Alon and Dubiner's result must satisfy*

$$c_d \geq 2^d 1.125^{\lfloor \frac{d}{3} \rfloor}.$$

If we take a fixed n and a large dimension d, then this implies:

Corollary 2. *For odd n and sufficiently large $d \geq d'$ the following lower bound holds:*

$$f(n,d) \geq 2.08^d.$$

For $n=3$ this is weaker than the existing bound $f(3,d) \geq 2.217389^d$ (for large d).

It is interesting to note that we have for the general case considered in the Theorem a simple and uniform proof based on only one particular value: $f(3,3)=19$. It is not surprising that this can be extended to $f(3,d)$, but it is certainly surprising that the proof can be extended to different n.

For the related problem, where $g(n,d)$ denotes the minimal number such that any set of $g(n,d)$ *distinct* vectors in $\mathbb{Z}_n^d$ has a zero sum of length n, we show similar bounds. It is known that $g(n,d) \geq (n-1)2^{d-1}+1$ for $n \geq 3$, and $g(n,d) \geq n2^{d-1}+1$ for even n, $g(3,3)=10$, $g(3,4)=21$, (see Harborth [14], Brenner [3], Frankl, Graham and Rödl [9], and Kemnitz [15], [16]). By the pigeonhole principle we have that $g(n,d) \geq \frac{f(n,d)-1}{n-1}+1$. Therefore the theorem immediately implies the following

Corollary 3. *Let $n \geq 3$ be an odd integer. The following bound holds:*

$$g(n,d) \geq 1.125^{\lfloor \frac{d}{3} \rfloor}2^d + 1.$$

In particular, for large d:

$$g(n,d) \geq 2.08^d.$$

For fixed d and large n the corollary is much weaker than the theorem. One trivially has that $n \le g(n,d)$. The lower bound of the corollary is independent of n and therefore in this respect not optimal. But for fixed n and large d the corollary above may be of interest.

2. Proof

In the proof of the theorem we start off from Harborth's example which shows that $f(3,3) \ge 19$. We first extend this in dimension 3 from $n=3$ to arbitrary n. We then increase the dimension by an explicit product construction.

We take the 9 vectors considered by Harborth but we consider these in $\mathbb{Z}_n^3$ (not only in $\mathbb{Z}_3^3$).

$$\begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}.$$

If $n \ge 3$, then these vectors are distinct in $\mathbb{Z}_n^3$. We shall prove:

Lemma. *Let $n \ge 3$ be an odd integer. If any n vectors taken from a multiset of the above 9 vectors add to $0 \in \mathbb{Z}_n^3$, then necessarily one has taken n times the very same vector.*

To avoid such kind of zero sum our example in $d=3$ starts off with $n-1$ copies of each of the above 9 elements in $\mathbb{Z}_n^3$. In dimension $d>3$ we construct our vectors as follows. Up to dimension $3\left\lfloor \frac{d}{3} \right\rfloor$ we take all $9^{\lfloor \frac{d}{3} \rfloor}$ vectors which are composed out of any of the 9 vectors in the coordinates $3i+1$ up to $3i+3$, where $i=0,\dots,\left\lfloor \frac{d-3}{3} \right\rfloor$. In the remaining $d_0=0,1$ or 2 coordinates we take all 2^{d_0} combinations of vectors composed of 0 and 1 only. Note that $2^{d_0} \times 2^{3\lfloor \frac{d}{3} \rfloor} = 2^d$. We take any of these vectors $n-1$ times such that the total number of used vectors is $1.125^{\lfloor \frac{d}{3} \rfloor} 2^d (n-1)$.

Suppose that there is a choice of n vectors which sums to the zero vector, modulo n. Any choice of n vectors which sums in the last d_0 coordinates to 0 or n must necessarily have n times the same value of $\varepsilon_i \in \{0,1\}$. Moreover, we use the lemma for triples of dimensions $(3i+1, 3i+2, 3i+3)$, where $i=0,\dots,\left\lfloor \frac{d}{3} \right\rfloor$. Suppose for a moment the lemma is proven: then a zero sum must take in these three coordinates n copies of the same entries. But since there are $n-1$ copies of each d-dimensional vector only, the n vectors cannot be identical. They differ in at least one of the three dimensions $3i+1$ up to

$3i+3$ (say) or in the last d_0 dimensions. In these dimensions there is no zero sum of length n, which proves the theorem.

It is therefore sufficient to prove the lemma. Suppose our choice of the n vectors uses a_i copies of the i-th vector, where $i = 1, \ldots, 9$ and where $0 \leq a_i \leq n-1$. We then have the following system of equations.

$$\begin{aligned}
&(1) && a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = n \\
&(2) && 2a_1 + a_6 + a_7 + a_8 + a_9 \equiv 0 \bmod n \\
&(3) && a_1 + a_4 + a_5 + a_8 + 2a_9 \equiv 0 \bmod n \\
&(4) && 2a_1 + a_3 + a_5 + a_7 + 2a_8 + 2a_9 \equiv 0 \bmod n.
\end{aligned}$$

We show first that the three expressions on the left hand side of (2), (3) and (4) must be equal to n. In order to see this, we must consider various cases.

Case 1) In equation (2) we must exclude that $2a_1 + a_6 + a_7 + a_8 + a_9 = 0$ or $\geq 2n$. Similarly, cases 2) and 3) will treat the equations (3) and (4).

Case 1a) Equation (1) and $0 \leq a_i < n$ shows that $2a_1 + a_6 + a_7 + a_8 + a_9 < 2n$.

Case 1b) So let us suppose for a contradiction that $2a_1 + a_6 + a_7 + a_8 + a_9 = 0$. This implies that $a_1 = a_6 = a_7 = a_8 = a_9 = 0$. Our system of equations simplifies to

$$\begin{aligned}
&(1') && a_2 + a_3 + a_4 + a_5 = n \\
&(3') && a_4 + a_5 \equiv 0 \bmod n \\
&(4') && a_3 + a_5 \equiv 0 \bmod n.
\end{aligned}$$

Moreover

$$\begin{aligned}
&(1') - (3') && a_2 + a_3 \equiv 0 \bmod n \\
&(1') - (4') && a_2 + a_4 \equiv 0 \bmod n.
\end{aligned}$$

If we assume that $a_4 + a_5 = n$, then $a_2 = a_3 = 0$, by (1′). But then by (1′) − (4′): $a_4 \equiv 0 \bmod n$. Then $0 \leq a_i < n$ shows that $a_4 = 0$, so that $a_5 = n$, a contradiction to $a_5 < n$. Similarly, if we assume that $a_2 + a_3 = n$, so that $a_4 = a_5 = 0$, then the same kind of contradiction follows from (1′) − (4′): $a_2 \equiv 0 \bmod n$, and $a_3 = n$.

Case 2a) Since $a_9 < n$ we can exclude that

$$(3) \qquad a_1 + a_4 + a_5 + a_8 + 2a_9 \geq 2n.$$

Case 2b) So, let us suppose that

$$(3) \qquad a_1 + a_4 + a_5 + a_8 + 2a_9 = 0,$$

which implies that $a_1=a_4=a_5=a_8=a_9=0$.

Our system of equations then is

$$\begin{aligned} &(1') & a_2+a_3+a_6+a_7&=n & \\ &(2') & a_6+a_7&=n & \text{by case 1)}\\ &(4') & a_3\qquad +a_7&\equiv 0 \bmod n. & \end{aligned}$$

This implies $a_2=a_3=0$. Then (4′) implies that $a_7\equiv 0 \bmod n$, i.e. $a_7=0$, which contradicts $a_6<n$.

Case 3a) Suppose that

$$(4) \qquad 2a_1+a_3+a_5+a_7+2a_8+2a_9\geq 2n.$$

Then 2(1) – (4) gives

$$2a_2+a_3+2a_4+a_5+2a_6+a_7\leq 0,$$

so that

$$a_2=a_3=a_4=a_5=a_6=a_7=0.$$

Our system of equations then is:

$$\begin{aligned} &(1') & a_1+a_8+\ \ a_9&=n & \\ &(2') & 2a_1+a_8+\ \ a_9&=n & \text{by case 1)}\\ &(3') & a_1+a_8+2a_9&=n & \text{by case 2).} \end{aligned}$$

This implies that $a_1=a_9=0$, $a_8=n$, which contradicts $a_8<n$.

Case 3b) So, we assume that

$$(4) \qquad 2a_1+a_3+a_5+a_7+2a_8+2a_9=0,$$

which implies

$$a_1=a_3=a_5=a_7=a_8=a_9=0.$$

By equation (2) we see that $a_6=n$, which contradicts $a_6<n$.

The above case study shows that the left hand sides of (1), (2), (3), (4) are all equal to n. Let us look at equation

$$(1)+(4)-(2)-(3) \qquad a_2+2a_3+a_5+a_7+a_8=0.$$

Because of $0\leq a_i$ we find that

$$a_2=a_3=a_5=a_7=a_8=0.$$

Therefore equation (4) simplifies to

$$2a_1+2a_9=n,$$

which is a contradiction, since n is odd, by assumption. ∎

Remarks. It seems conceivable that starting off from other examples for small fixed n and d, one might be able to improve the lower bound. But it is not at all obvious that this will work for any particular value of $f(n,d)$. It is indeed somewhat surprising that an argument modulo 3 can be extended to an argument modulo all odd integers.

For even n the lower bound cannot hold in general since for $n=2^a$ one knows that $f(2^a,d)=(2^a-1)2^d+1$.

The author would like to thank N. Alon, T. Bekehermes, I. Ruzsa, the anonymous referees, and in particular S. Kubertin for remarks on an earlier version of this manuscript.

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