TRIPLES OF PRIMES IN ARITHMETIC PROGRESSIONS

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Abstract

We show that there exist sets of primes $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P} \cap [1, N]$ with $|\mathcal{A}| = s$, $|\mathcal{B}| = t$ such that all $\frac{1}{2}(a_i + b_j)$ are also prime, and where $s \geqslant 0.33^t N/(\log N)^{t+1}$ holds, for sufficiently large N.

Grosswald [**4**] considered the number of triples of primes in arithmetic progression:

$$G_3(N) := \left| \left\{ (p_1 < p_2 < p_3) : p_3 \leqslant N,\ p_1, p_2, p_3 \in \mathcal{P} \text{ and } p_2 = \frac{p_1 + p_3}{2} \right\} \right|,$$

where $\mathcal{P}$ denotes the set of primes. Grosswald proved that the following asymptotic expansion holds:

$$G_3(N) = \frac{1}{2} C \frac{N^2}{(\log N)^3} \left\{ 1 + \sum_{j=1}^{r} \frac{a_j}{(\log N)^j} + O\left(\frac{1}{(\log N)^{r+1}} \right) \right\},$$

where $C = \prod_{p>2} \left(1 - 1/(p-1)^2\right) = 0.6601618\ldots$ is the twin primes constant, and where the a_j are computable constants.

In a different additive problem involving primes, Pomerance, Sárközy, and Stewart [**9**] proved that for sufficiently large N and $t < \log N$ there exist sets of integers $\mathcal{A}, \mathcal{B} \subseteq [1, N]$ with $|\mathcal{B}| = t$ and

$$|\mathcal{A}| > \frac{N}{t(\log N)^t}$$

such that $\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\} \subseteq \mathcal{P}$. In particular, they deduced for large N and arbitrary $\varepsilon > 0$ that there exist such sets $\mathcal{A}, \mathcal{B}$ with

$$|\mathcal{A}|, |\mathcal{B}| \geqslant (1 - \varepsilon) \frac{\log N}{\log \log N}. \tag{1}$$

In this paper we combine the two approaches and show the following results.

THEOREM 1 *Let N be sufficiently large. For $t \geqslant 2$ there exist disjoint sets of primes $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P} \cap [1, N]$, with $|\mathcal{B}| = t$ and $|\mathcal{A}| = s \geqslant 0.33^t N/(\log N)^{t+1}$, such that all $\frac{1}{2}(a_i + b_j)$ are also prime.*

In particular, this has an implication for sets of equal size.

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THEOREM 2 *For large N there exist disjoint sets of primes $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P} \cap [1, N]$ with*

$$|\mathcal{A}|, |\mathcal{B}| \geqslant \frac{\log N}{\log\log N} - \frac{10}{9}\frac{\log N}{(\log\log N)^2}$$

and where all $\frac{1}{2}(a_i + b_j)$ are prime.

For the proof of Theorem 1 we use Grosswald's theorem [**4**]. This will be combined with a counting argument. A very nice and convenient way of counting is based on a result from extremal graph theory and was described by Gyarmati [**6**], who constructed large square-free sumsets. The main difference from our situation is that the square-free integers have a positive density and that the density of square-free integers is considerably higher on certain arithmetic progressions. In our present application we adapt this method to thinner sequences.

We use the following lemma due to Kővari, Sós and Turán [**8**]. (Compare [**2**, Theorem IV.10] of Bollobás.)

LEMMA 1 *Let $G(V_1 \times V_2, E)$ denote a bipartite graph with m vertices in the first class V_1 and n vertices in the second class V_2. Let $z(m, n; s, t)$ denote the maximal number of edges of G such that G does not contain a complete bipartite graph $K_{s,t}$ with s vertices in the first class and t in the second. Then, for all natural numbers m, n, s and t we have*

$$z(m, n; s, t) \leqslant s^{1/t} n m^{1-1/t} + tm.$$

Proof of Theorem 1. We define the bipartite graph $G(V_1 \times V_2, E)$ as follows: the sets of vertices are $V_1 = V_2 = \mathcal{P} \cap [1, N]$; the set of edges is

$$E = \left\{ (v_1, v_2) \in V_1 \times V_2 \mid v_1 \neq v_2 \text{ and } \frac{v_1 + v_2}{2} \in \mathcal{P} \right\}.$$

So, an edge corresponds to a triple (p_1, p_2, p_3) with $p_2 = \frac{1}{2}(p_1 + p_3)$. Note that $p_1 = p_2 = p_3$ is not allowed. A complete bipartite graph $K_{s,t}$ corresponds to disjoint sets of primes $\mathcal{A}$ and $\mathcal{B}$ of sizes s and t such that all $\frac{1}{2}(a_i + b_j)$ are also prime. Grosswald's theorem says that this graph G contains many edges. The Kővari–Sós–Turán theorem says that a bipartite graph with many edges must have a large $K_{s,t}$ as a subgraph.

Suppose that G does not contain a complete bipartite graph $K_{s,t}$, where the first class of $K_{s,t}$ lies in the first class V_1 of G and the second class in V_2. We then find by the lemma that

$$0.3307 \frac{N^2}{(\log N)^3} \leqslant |E| \leqslant s^{1/t} \pi(N)^{2-1/t} + t\pi(N).$$

Here $\pi(N)$ denotes the number of primes less than N. By the prime number theorem we know that

$$\pi(N) = \frac{N}{\log N} + \frac{N}{(\log N)^2} + O\left(\frac{N}{(\log N)^3}\right)$$

so that in particular for large N we have the estimate $\pi(N) \geqslant N/\log N$. This implies for $t = O(N^{\varepsilon})$ that

$$\begin{aligned} \frac{0.3307}{\log N} &\leqslant s^{1/t} \frac{(\log N)^{1/t}}{N^{1/t}} + t \frac{\log N}{N} \\ &\leqslant s^{1/t} \frac{(\log N)^{1/t}}{N^{1/t}} \left(1 + \frac{1}{N^{1/3}}\right) \end{aligned}$$

and therefore

$$s \geqslant 0.3306^t \frac{N}{(\log N)^{t+1}}.$$

Note that since we can assume without loss of generality that $s \geqslant t$ the choice $t \leqslant O(N^\varepsilon)$ is not restrictive. Hence for s smaller than the above bound the graph G contains a complete bipartite graph $K_{s,t}$ which proves Theorem 1.

For Theorem 2 we want both sets to be of the same size, that is, $s = t$. An easy computation shows that

$$t = \left\lfloor \frac{\log N}{\log\log N} - \frac{10}{9}\frac{\log N}{(\log\log N)^2} \right\rfloor$$

is an admissible value, for sufficiently large N, since $\frac{10}{9} + \log 0.33$ is positive.

Grosswald [**4**] also considered the number $G_k(N)$ of prime k-tuples in arithmetic progression. However, in this case, he was only able to prove a corresponding formula $G_k(N) \sim C_k N/(\log N)^k$ subject to a strong version of a conjecture of Hardy and Littlewood [**7**]. Unconditionally it is unknown whether there are infinitely many quadruples of primes in arithmetic progression. It is of course possible to adapt the counting argument to deal with the conjectured asymptotic formula for $G_k(N)$, but we refrain from stating such conditional results.

Background material may be found in [**1**, **3**, **5**].

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