ON CLUSTER PRIMES

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Abstract. In this note we prove a conjecture by Blecksmith, Erdős, and Selfridge on the number of cluster primes. A prime p is said to be a cluster prime if every positive even integer $2r \leq p-3$ can be written as a difference of two primes, $2r = q - q'$, where $q' \leq q \leq p$. Let $\pi_C(x)$ denote the number of cluster primes below x. Then the following estimate holds:

$$\pi_C(x) = O\left(\frac{x}{\exp(\frac{1}{60}(\log\log x)^2)}\right).$$

1. Introduction

Blecksmith, Erdős and Selfridge [1] defined a prime $p > 2$ to be a cluster prime if every positive even integer $2r \leq p-3$ can be written as a difference of two primes, $2r = q - q'$, where $q' \leq q \leq p$. It is an open question, whether there exists infinitely many cluster primes. Guy ([3], section C1) attributes this question to Erdős.

Blecksmith et. al. [1] proved that the counting function $\pi_C(x)$ of cluster primes can be bounded from above: for all positive s,

$$\pi_C(x) = O_s\left(\frac{x}{(\log x)^s}\right).$$

It is the purpose of this note to prove a better bound, i.e. that cluster primes are rare. This new bound was indeed conjectured by Blecksmith et. al. [1].

Theorem. *The number $\pi_C(x)$ of cluster primes below x is bounded by*

$$\pi_C(x) = O\left(\frac{x}{\exp(\frac{1}{60}(\log\log x)^2)}\right).$$

As Blecksmith, Erdős and Selfridge show, the problem is related to the prime k-tuple conjecture. It is proved that for a cluster prime p the interval $[p-t, p)$ must contain sufficiently many primes, which explains the name cluster prime. This allows us to apply an upper bound sieve. In Blecksmith et.al. [1], Brun's version of the small sieve is used. The principal problem is that the authors arrive at a constant M, of which the dependence on the sieve dimension s is not at all clear. This prohibits taking an increasing s.

Filaseta [2] mentioned that an application of Hooley's almost pure sieve proves the result with $s = \varepsilon \log\log\log x$, thus obtaining an upper bound of $\pi_C(x) = O\left(\frac{x}{\exp(a \operatorname{loglog} x \operatorname{logloglog} x)}\right)$ for some positive constant a.

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In this note we apply the large sieve method, due to Montgomery [4]. In fact, we make use of the following lemma due to Vaughan [6], which is an elaborated version of the large sieve method, perfectly fitting to our application.

Lemma 1 (Montgomery [4], Vaughan [6])**.** *Let $\mathcal{P}$ denote the set of primes and let $\omega : \mathcal{P} \to \mathbb{N}$ with $0 \le \omega(p) \le p-1$. Let $\mathcal{A} \subset [1,N]$ denote a set of integers which lies outside $\omega(p)$ residue classes modulo the prime p. Then the number $A(x)$ of elements $n \in \mathcal{A}, n \le x$ satisfies*

$$A(x) \le \frac{2x}{L}, \text{ where } L = \sum_{q \le x^{1/2}} \mu^2(q) \prod_{p|q} \frac{\omega(p)}{p-\omega(p)}.$$

Moreover, for L the following estimate holds:

$$L \ge \max_{m \in \mathbb{N}} \exp\left(m \log \left(\frac{1}{m} \sum_{p \le x^{1/(2m)}} \frac{\omega(p)}{p} \right) \right).$$

2. Proof of the Theorem

If p is a cluster prime, then the even integers like $p-9$ or $p-15$ are the difference of two primes with $q, q' \le p$. In particular, there must be a prime in the interval $[p-6, p)$. More generally, an even integer $2r \in [p-t, p-3]$ must be represented by a prime $q \in [p-t, p]$ and a prime $q' \in [1, t]$. By the prime number theorem the number of primes in $[1,t]$ is $(1+o(1))\frac{t}{\log t}$. We see that for any $\varepsilon > 0$ there must be at least $s := (\frac{1}{2} - \varepsilon)\log t$ primes in $[p-t, p)$.

Since the average gap between primes of size x is about $\log x$ we see that this is a useful criterion, for $t = O\left((\log x)^\delta\right)$ (with $0 < \delta < 1$). On the contrary, for sufficiently large t one expects that an interval of length t has about $\frac{t}{\log t}$ primes so that this criterion becomes useless.

There are (trivially) at most $\binom{t}{s}$ possibilities to place s primes in an interval of length t. For any pattern of s primes in $[p-t, p)$ we will give an upper bound on the number of prime s-tuples below x. This bound will not depend on the particular pattern. So, multiplying this bound by the upper bound for the number of patterns, $\binom{t}{s}$, gives an upper bound on the number of $p \le x$ such that the interval $[p-t, p)$ contains (at least) s primes.

We prove the following lemma:

Lemma 2. *Let $\delta = \frac{1}{2e}$ and $t = (\log x)^\delta$. Let ε be a sufficiently small positive constant. Let $A(x)$ denote the number of integers $n \le x$ such that the interval $[n-t, n)$ contains at least $s = \left(\frac{1}{2} - \varepsilon\right)\log t$ primes. Then the following estimate holds:*

$$A(x) = O\left(\frac{x}{\exp\left((\frac{1}{4e^2} - \varepsilon)(\log\log x)^2 \right)} \right).$$

We fix a particular pattern $a_1 < a_2 < \cdots < a_s$. If all $n - a_i$ are prime simultaneously, then the integers n avoid the residue classes $a_i \bmod p$ for $p \le n-t$. If $p > t$, then the number of forbidden classes is $\omega(p) = s$.

We choose $m = \lceil \delta^2 (\log\log x)^2 \rceil$, where x is large. With $y = x^{1/(2m)}$ we have $\log\log y = \log\log x - 2\log\log\log x + O(1)$. So we find that

$$\begin{aligned}\sum_{p\le y} \frac{\omega(p)}{p} &\ge \sum_{t\le p\le y} \frac{\omega(p)}{p}\\ &\ge s(\log\log y - \log\log t + o(1))\\ &\ge (\frac{1}{2} - \varepsilon)\delta(\log\log x)(\log\log x - 3\log\log\log x + O(1))\\ &\ge (\frac{1}{2} - 2\varepsilon)\delta(\log\log x)^2.\end{aligned}$$

This implies the estimate

$$\begin{aligned}L &\ge \exp\left(\lceil \delta^2(\log\log x)^2\rceil \log\left(\frac{1}{\lceil \delta^2(\log\log x)^2\rceil}\delta(\frac{1}{2} - 2\varepsilon)(\log\log x)^2\right)\right)\\ &\ge \exp\left(\delta^2(\log\log x)^2 \log\left(\frac{1}{\delta}\left(\frac{1}{2} - 3\varepsilon\right)\right)\right)\\ &\ge \exp\left(\delta^2(\log\log x)^2 \log(e - 6e\varepsilon)\right)\\ &\ge \exp\left(\delta^2(1-7\varepsilon)(\log\log x)^2\right)\\ &\ge \exp\left(\left(\frac{1}{4e^2} - \varepsilon\right)(\log\log x)^2\right).\end{aligned}$$

Therefore, for any fixed pattern $a_1 < a_2 < \cdots < a_s$ there are at most

$$\frac{2x}{\exp((\frac{1}{4e^2} - \varepsilon)(\log\log x)^2)}$$

values $n \le x$ such that all $n - a_i$ are prime. Thus the lemma is proved.

To prove the theorem we only need to recall that

$$\pi_C(x) \le \binom{t}{s} \frac{2x}{\exp(\delta^2(1-7\varepsilon)(\log\log x)^2)}.$$

Because of

$$\binom{t}{s} \le t^s \le \exp\left((\frac{1}{2} - \varepsilon)\delta^2(\log\log x)^2\right)$$

we find that

$$\pi_C(x) = O\left(\frac{x}{\exp((\frac{1}{8e^2} - \varepsilon)(\log\log x)^2)}\right).$$

3. Further comments

No serious attempt has been made at optimizing the constant $\frac{1}{60}$ or $\frac{1}{8e^2} - \varepsilon$ that appears in the theorem. Some improvement is possible. We only mention the following: Vaughan's argument in Lemma 1 can be refined to

$$L \ge \max_{m\in\mathbb{N}} \exp\left(m \log\left(\frac{e - \varepsilon_m}{m} \sum_{p \le x^{1/(2m)}} \frac{\omega(p)}{p}\right)\right).$$

Here the ε_m are positive constants that tend to 0 as m goes to infinity. This allows to use $c_1 \approx \frac{\delta}{2}$ and $\delta \approx \frac{1}{2}$ and proves the theorem with $\frac{1}{8} - \varepsilon$ instead of $\frac{1}{60}$.

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