A Short Survey on Upper and Lower Bounds for Multidimensional Zero Sums

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After giving some background on sums of residue classes we explained the following problem on multidimensional zero sums which is well known in combinatorial number theory:

Let $f(n,d)$ denote the least integer such that any choice of $f(n,d)$ elements in $\mathbb{Z}_n^d$ contains a subset of size n whose sum is zero. Harborth [12] proved that $(n-1)2^d + 1 \le f(n,d) \le (n-1)n^d + 1$. The lower bound follows from the example in which there are $n-1$ copies of each of the 2^d vectors with entries 0 or 1. The upper bound follows since any set of $(n-1)n^d + 1$ vectors must contain, by the pigeonhole principle, n vectors which are equivalent modulo n.

If d is fixed, the upper bound was improved considerably by Alon and Dubiner [2] to $c_d\, n$. Erdős, Ginzburg, and Ziv [6] proved that $f(n,1) = 2n-1$ and Kemnitz conjectured that $f(n,2) = 4n-3$. There are partial results due to Kemnitz [14], as well as Gao [8], [9], [10], [11], Rónyai [16] and Thangadurai [17].

For example, Rónyai [16] proved that for primes p one has $f(p,2) \le 4p-2$, which implies that $f(n,2) \le 4.1\, n$. Gao [11] extended this to powers of primes: $f(p^a,2) \le 4p^a - 2$.

If n is fixed but d is increasing not very much is known.

$f(2^a,d) = (2^a-1)2^d+1$	see Harborth [12],
$f(3,3) = 19$	see Harborth [12], Brenner [3],
$f(3,4) = 41$	see Brown and Buhler [4], Brenner [3], Kemnitz [14],
$91 \le f(3,5) \le 121$	see Kemnitz [13],
$f(3,18) \ge 300 \times 2^{12}$	see Frankl, Graham, Rödl [7],
$f(3,d) \ge 2.179^d$ for $d \ge d'$	see Frankl, Graham, Rödl [7],
$f(n,d) \ge (1.125)^{\lfloor \frac{d}{3} \rfloor} (n-1)2^d + 1$	for odd n, see Elsholtz [5],
$f(n,d) = o(n^d)$	if n is fixed and d goes to infinity, see Alon and Dubiner,
$f(3,d) = O(\frac{3^d}{d})$	see Meshulam [15] .

The proof of our lower bound was the first nontrivial lower bound for any value of $f(n,d)$ with $n > 3$. It is based on a set of 9 vectors considered by Harborth [12] tp prove that $f(3,3) = 19$, but we consider these in $\mathbb{Z}_n^3$ (not only in $\mathbb{Z}_3^3$).

$$\begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}.$$

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We proved:
Let $n \geq 3$ be odd. If any n vectors taken from a multiset of the above 9 vectors add to $0 \in \mathbb{Z}_n^3$, then necessarily one has taken n times the very same vector. Therefore, taking $n-1$ copies of each vector only avoids zero sums of length n.

Remarks
It seems conceivable that starting off from other examples for small fixed n and d, one might be able to improve the lower bound. But it is not at all obvious that this will work for any particular value of $f(n,d)$. The proof in Elsholtz [5] is in principle elementary, and might be accessible to generalization, possibly with the help of computers. Albeit, an exhaustive search to determine values of $f(n,d)$ seems out of reach even for moderate sized n and d.

I hope that this survey of existing results convinces the reader that there is still a huge gap between the known upper and lower bounds.

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