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Maximal Dimension of Unit Simplices

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Abstract. For an arbitrary field $\mathbb{F}$ the maximal number $\omega(\mathbb{F}^n)$ of points in $\mathbb{F}^n$ mutually distance 1 apart with respect to the standard inner product is investigated. If the characteristic char($\mathbb{F}$) is different from 2, then the values of $\omega(\mathbb{F}^n)$ lie between $n-1$ and $n+2$. In particular, we answer completely for which n a simplex of q points with edge length 1 can be embedded in rational n-space. Our results imply for almost all even n that $\omega(\mathbb{Q}^n) = n$ and for almost all odd n that $\omega(\mathbb{Q}^n) = n - 1$.

1. Introduction

1.1. *Motivation*

It is well known that the unit triangle cannot be embedded into $\mathbb{Q}^2$. In $\mathbb{Q}^8$ we easily find the regular unit simplex spanned by the following nine points:

$$\begin{pmatrix} \frac{1}{2} \\ \pm\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \pm\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \pm\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \pm\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

In $\mathbb{R}^n$ we can obviously place exactly $n+1$ points defining a unit simplex. If we define $\omega(\mathbb{F}^n)$ to be the maximal number of points mutually at unit distance over the field $\mathbb{F}$, then $\omega(\mathbb{R}^n) = n + 1$. (For a precise definition of the distance function see the next subsection.) On the other hand, $\omega(\mathbb{Q}^n)$ had previously only been determined up to the solution of a diophantine equation, see [4]. In this paper we solve this diophantine equation and thus determine $\omega(\mathbb{Q}^n)$ completely. It turns out that the prime factorizations of n and $n+1$ are an important ingredient in the classification. A consequence of this result is that $\omega(\mathbb{Q}^n) = n$ for almost all even n and $\omega(\mathbb{Q}^n) = n - 1$ for almost all odd n.

The problem of determining $\omega(\mathbb{F}^n)$ is loosely connected to the problem of the chromatic number of $\mathbb{F}^n$. Let $\chi(\mathbb{F}^n)$ be the minimum number of colours needed to colour $\mathbb{F}^n$ so that no two points of the same colour are one unit apart. Even the determination of the chromatic number of the plane is a major outstanding problem. It is only known that $4 \leq \chi(\mathbb{R}^2) \leq 7$. For a survey see [10] and [11]. We may interpret $\omega(\mathbb{F}^n)$ as the clique number of the *unit-distance graph* $G_1(\mathbb{F}^n)$, which has vertex set $\mathbb{F}^n$ with vertices $x,\ y$ connected by an edge, if they are at unit distance, see [5]. Then $\omega(\mathbb{F}^n)$ is a lower bound for the chromatic number $\chi(G_1(\mathbb{F}^n)) = \chi(\mathbb{F}^n)$. For $\mathbb{F} = \mathbb{Q}$ some exact values are known: $\chi(\mathbb{Q}^2) = \chi(\mathbb{Q}^3) = 2,\ \chi(\mathbb{Q}^4) = 4$ [1], [12], $\chi(\mathbb{Q}^5) \geq 7$ [8], see also [13]. This compares very well with $\omega(\mathbb{Q}^2) = \omega(\mathbb{Q}^3) = 2$ and $\omega(\mathbb{Q}^4) = \omega(\mathbb{Q}^5) = 4$. While $\omega(\mathbb{Q}^n) \leq \omega(\mathbb{R}^n) = n + 1$, the chromatic numbers $\chi(\mathbb{Q}^n)$ and $\chi(\mathbb{R}^n)$ increase exponentially, see [6] and [9].

1.2. *Results*

As many of our arguments hold for arbitrary fields, we start with a rather general setting. The most interesting applications are perhaps to subfields of $\mathbb{R}$.

Let $\mathbb{F}$ be an arbitrary field, let $x = (x_i) \in \mathbb{F}^n$ and let $y = (y_i) \in \mathbb{F}^n$, n a positive integer. Define the *quadratic distance* of points x and y by the standard inner product:

$$\Delta(x, y) = (x - y)^2 = \sum_{i=1}^{n}(x_i - y_i)^2.$$

By $\omega(\mathbb{F}^n)$ we denote the maximal number of points in $\mathbb{F}^n$, which mutually have quadratic distance 1.

We call the standard inner product of $\mathbb{F}^n$ *nonisotropic* if

$$x \cdot x = 0 \iff x = 0 \qquad \text{for every} \quad x \in \mathbb{F}^n.$$

Observe that, e.g., for $\mathrm{char}(\mathbb{F}) = 2$ and $n \geq 2$ the standard inner product is not nonisotropic, while it is nonisotropic for real fields.

In case of a nonisotropic inner product the next theorem completely determines $\omega(\mathbb{F}^n)$ up to the solution of a quadratic equation in the very last case, with values bounded by $n - 1 \leq \omega(\mathbb{F}^n) \leq n + 2$. For the proof we combine methods from design theory (Bruck–Ryser type arguments, see [3]) with those from linear algebra and number theory.

Theorem 1.

(A) *If* char$(\mathbb{F}) = 2$, *then* $\omega(\mathbb{F}^n) = 2$ *for every* $n \geq 1$.
(B) *If* char$(\mathbb{F}) \neq 2$ *and n is even, then the following statements hold true*:
  (1) *If* $\sqrt{n+1} \notin \mathbb{F}$, *then* $\omega(\mathbb{F}^n) = n$.
  (2) *Suppose that* $\sqrt{n+1} \in \mathbb{F}$.
  If $n = -2$ *in* $\mathbb{F}$, *then* $\omega(\mathbb{F}^n) = n+2$, *otherwise* $\omega(\mathbb{F}^n) = n+1$.
(C) *If* char$(\mathbb{F}) \neq 2$ *and n is odd, then the following statements hold true*:
  (1) *Suppose that* $\sqrt{(n+1)/2} \in \mathbb{F}$.
  If $n = -2$ *in* $\mathbb{F}$, *then* $\omega(\mathbb{F}^n) = n+2$, *otherwise* $\omega(\mathbb{F}^n) = n+1$.
  (2) *Suppose that* $\sqrt{(n+1)/2} \notin \mathbb{F}$.
  If the equation

$$u^2 \;+\; 2(n-1)v^2 \;=\; n \tag{1}$$

  has a solution with $u \in \mathbb{F}$ *and* $v \in \mathbb{F}$, *then* $\omega(\mathbb{F}^n) = n$.
  If the standard inner product of $\mathbb{F}^n$ *is nonisotropic and if equation* (1) *is unsolvable in* $\mathbb{F}$, *then* $\omega(\mathbb{F}^n) = n-1$.

Theorem 1 immediately implies the following corollary:

Corollary 1. *The smallest field* $\mathbb{F}$ *over* $\mathbb{Q}$ *such that* $\omega(\mathbb{F}^n) = n+1$ *for every positive integer n is* $\mathbb{F} = \mathbb{Q}[\sqrt{2},\ \sqrt{3},\ \sqrt{5},\ \sqrt{7}, \ldots]$.

The next theorem describes the complete evaluation of $\omega(\mathbb{Q}^n)$.

Theorem 2.

(A) *Let n be even.*
  If $n+1$ *is the square of an integer, then* $\omega(\mathbb{Q}^n) = n+1$, *otherwise* $\omega(\mathbb{Q}^n) = n$.
(B) *Let n be odd.*
  (1) *If* $(n+1)/2$ *is the square of an integer, then* $\omega(\mathbb{Q}^n) = n+1$.
  (2) *Suppose that* $(n+1)/2$ *is not the square of an integer. Let* $n = n_1 n_2^2$ *be the factorization of n with a unique squarefree divisor* n_1. *If* n_1 *has no prime divisor* $p \equiv \pm 3$ *modulo* 8, *then* $\omega(\mathbb{Q}^n) = n$, *and* $\omega(\mathbb{Q}^n) = n-1$ *otherwise.*

Let $N_{\text{even}}(t)$ denote the number of even integers $n \leq t$ with $\omega(\mathbb{Q}^n) \neq n$ and $N_{\text{odd}}(t)$ the number of odd integers $n \leq t$ with $\omega(\mathbb{Q}^n) \neq n-1$. Evidently, part A of Theorem 2 implies $N_{\text{even}}(t) \sim \sqrt{t}/2$. Landau's method [7, Sections 177–183] on estimating the number of positive integers $\leq t$ with specified prime factors only, allows us to conclude from part B of Theorem 2 that we have $N_{\text{odd}}(t) \sim ct/(\log t)^{1/2}$ for some positive constant c. So we may state the following consequence.

Corollary 2.

(1) $\omega(\mathbb{Q}^n) = n$ *for almost all even integers n.*
(2) $\omega(\mathbb{Q}^n) = n-1$ *for almost all odd integers n.*

Theorem 2 and Corollary 2 have the following geometric interpretation. For those n with $\omega(\mathbb{Q}^n) = n+1$ it is possible to rotate the regular n-dimensional simplex with edge length 1 in $\mathbb{R}^n$ so that all coordinates of the $n+1$ vertices become rational. For almost all dimensions n, however, this type of rotation does not exist and only a unit simplex with up to n (for even n), respectively $n-1$ (for odd n), vertices can be embedded into $\mathbb{Q}^n$.

2. Proof of Theorem 1

First we settle the case char$(\mathbb{F}) = 2$ and establish the upper bounds. Let $x_0, \dots, x_m$ be $m+1$ points in $\mathbb{F}^n$ mutually at quadratic distance 1. We may suppose $x_0 = 0$. Then the other vectors $x_j,\ j \geq 1$, have unit length and for $i, j \geq 1,\ i \neq j$, we have

$$(x_i - x_j)^2 = x_i^2 - 2x_ix_j + x_j^2 = 1, \qquad 2x_ix_j = 1.$$

If char$(\mathbb{F}) = 2$ and $m \geq 2$, then the last equation implies a contradiction. This proves part A.

From now on we assume char$(\mathbb{F}) \neq 2$. Let the $(m \times n)$-matrix A be formed by the rows $x_1, \dots, x_m$. Then AA^T is an $(m \times m)$-matrix with entries 1 in the main diagonal and $\frac{1}{2}$ in the other positions so that

$$\det AA^T = (m+1)\frac{1}{2^m}. \tag{2}$$

If $m \neq -1$ in $\mathbb{F}$, then $\det AA^T \neq 0$, so that

$$\text{rank } AA^T = m \leq \text{ rank } A \leq n. \tag{3}$$

For $m = n+1$ and $n \neq -2$ in $\mathbb{F}$ inequality (3) leads to a contradiction, which implies

$$\omega(\mathbb{F}^n) \leq n+1, \qquad \text{if} \quad n \neq -2 \quad \text{in } \mathbb{F}. \tag{4}$$

If $n = -2$ in $\mathbb{F}$, then a contradiction in (3) occurs for $m = n+2$, which shows

$$\omega(\mathbb{F}^n) \leq n+2, \qquad \text{if} \quad n = -2 \quad \text{in } \mathbb{F}. \tag{5}$$

Let $m = n$. Then (2) may be written as

$$n+1 = 2^n(\det A)^2.$$

If n is even, then $n+1$ is the square of an element in $\mathbb{F}$, i.e. $\sqrt{n+1} \in \mathbb{F}$. If n is odd, then

$$(n+1)/2 = 2^{n-1}(\det A)^2$$

is the square of an element in $\mathbb{F}$, i.e. $\sqrt{(n+1)/2} \in \mathbb{F}$. So we know

$$\omega(\mathbb{F}^n) \leq n, \qquad \text{if} \quad \begin{cases} n \text{ is even and } \sqrt{n+1} \notin \mathbb{F}, \\ n \text{ is odd and } \sqrt{(n+1)/2} \notin \mathbb{F}. \end{cases} \tag{6}$$

To establish lower bounds we start the constructive part of our proof. We define a system σ of n points $x^{(1)}, \dots, x^{(n)}$ if n is even, and of $n-1$ points $x^{(1)}, \dots, x^{(n-1)}$ if n

is odd. If we take the vectors of σ as the rows of a matrix, this matrix has the following shape:

$$\left(\begin{array}{ccccc|ccccc}
\frac{1}{2} & & & & & & & & & \frac{1}{2} \\
\frac{1}{2} & & & & & & & & & -\frac{1}{2} \\
& \frac{1}{2} & & & & & & & \frac{1}{2} & \\
& \frac{1}{2} & & & & & & & -\frac{1}{2} & \\
& & \vdots & & & & & \vdots & & \\
& & & \vdots & & & \vdots & & & \\
& & & & \frac{1}{2} & \frac{1}{2} & & & & \\
& & & & \frac{1}{2} & -\frac{1}{2} & & & &
\end{array}\right).$$

Positions left empty have to be filled with zeros. If n is odd, a last column consisting of zeros only has to be added. Clearly, the points in σ are mutually at distance 1. Thus we have $\omega(\mathbb{F}^n) \geq n$ if n is even and $\omega(\mathbb{F}^n) \geq n-1$ if n is odd.

We try to extend the system σ equidistantly by a point $y = (y_1, \dots, y_n) \in \mathbb{F}^n$.

First we suppose that n is even.

The additional point y must have quadratic distance 1 to all points $x^{(1)}, \dots, x^{(n)}$ of σ, which for $m = 1, \dots, n/2$ implies

$$\Delta(y, x^{(2m-1)}) = (y_m - \tfrac{1}{2})^2 + (y_{n-m+1} - \tfrac{1}{2})^2 + \sum_{i \neq m, n-m+1} (y_i)^2 \ = \ 1,$$

$$\Delta(y, x^{(2m)}) = (y_m - \tfrac{1}{2})^2 + (y_{n-m+1} + \tfrac{1}{2})^2 + \sum_{i \neq m, n-m+1} (y_i)^2 \ = \ 1.$$

Taking the difference of these equations yields

$$y_{n-m+1} = 0 \qquad \text{for} \quad m = 1, \dots, \frac{n}{2}.$$

Now we have for $1 \leq m \leq n/2 - 1$,

$$\Delta(y, x^{(2m-1)}) = (y_m - \tfrac{1}{2})^2 + \tfrac{1}{4} + \sum_{i \neq m} (y_i)^2 = 1,$$

$$\Delta(y, x^{(2m+1)}) = (y_{m+1} - \tfrac{1}{2})^2 + \tfrac{1}{4} + \sum_{i \neq m+1} (y_i)^2 = 1.$$

Again subtracting equations leads to

$$y_{m+1} = y_m \qquad \text{for} \quad m = 1, \dots, \frac{n}{2} - 1.$$

Thus we see that y must have the form

$$y = y(s) = (\underbrace{s, \dots, s}_{n/2 \text{ entries}}, \underbrace{0, \dots, 0}_{n/2 \ \text{ entries}}), \qquad s \in \mathbb{F}.$$

Indeed, in this shape y has the same distance to all points $x^{(j)},\ j \le n$.

$$\Delta(y, x^{(j)}) = (s - \tfrac{1}{2})^2 + \left(\frac{n}{2} - 1\right) s^2 + \tfrac{1}{4} = 1,$$

$$\frac{n}{2} s^2 - s - \tfrac{1}{2} = 0. \tag{7}$$

If $n = 0$ in $\mathbb{F}$, then σ can be extended equidistantly by $y(-\frac{1}{2})$ to a system of $n+1$ points. According to (4) this is optimal, hence $\omega(\mathbb{F}^n) = n + 1$. Notice that $\sqrt{n+1} = 1 \in \mathbb{F}$ in this case, which corresponds to part B(2) of Theorem 1.

If $n \ne 0$ in $\mathbb{F}$, then we solve (7) for s:

$$s = \frac{1}{n}(1 \pm \sqrt{n+1}).$$

An equidistant extension of σ exists if and only if $\sqrt{n+1} \in \mathbb{F}$. If $\sqrt{n+1} \in \mathbb{F}$ and $n \ne -2$ in $\mathbb{F}$, then $\omega(\mathbb{F}^n) = n + 1$ according to (4).

To finish the case of even n, suppose now $n = -2$ in $\mathbb{F}$ and $\sqrt{n+1} = \sqrt{-1} \in \mathbb{F}$. The only candidates for an equidistant extension of σ are

$$y^{(1)} = y(s_1) \qquad \text{with} \quad s_1 = \frac{1}{n}(1 + \sqrt{n+1}) = -\tfrac{1}{2}(1 + \sqrt{-1}),$$

$$y^{(2)} = y(s_2) \qquad \text{with} \quad s_2 = \frac{1}{n}(1 - \sqrt{n+1}) = -\tfrac{1}{2}(1 - \sqrt{-1}).$$

The quadratic distance of $y^{(1)}$ and $y^{(2)}$ is

$$\Delta(y^{(1)}, y^{(2)}) = \frac{n}{2}(s_1 - s_2)^2 = -(\sqrt{-1})^2 = 1,$$

which means that in this case the system σ can be extended equidistantly to a system of $n + 2$ points. According to (5) no further extension is possible, $\omega(\mathbb{F}^n) = n + 2$.

We now consider the case of odd n.

In this case the system σ consists of $n - 1$ points $x^{(1)}, \ldots, x^{(n-1)}$. Again we try to extend σ equidistantly by a point $y = (y_1, \ldots, y_n)$. As above, the distance conditions

$$\Delta(y, x^{(j)}) = 1 \qquad \text{for} \quad j = 1, \ldots, n-1$$

force y to take the following form:

$$y = y(s, v) = (\underbrace{s, \ldots, s}_{(n-1)/2 \text{ entries}}, \underbrace{0, \ldots, 0,}_{(n-1)/2 \text{ entries}} v), \qquad s \in \mathbb{F}, \quad v \in \mathbb{F}.$$

Indeed, in this shape y has the same distance to all points $x^{(j)},\ j < n$.

$$\Delta(y, x^{(j)}) = (s - \tfrac{1}{2})^2 + \left(\frac{n-1}{2} - 1\right) s^2 + \tfrac{1}{4} + v^2 = 1,$$

$$\frac{n-1}{2} s^2 - s + v^2 - \tfrac{1}{2} = 0. \tag{8}$$

An equidistant extension of σ exists if and only if (8) has a solution with $s \in \mathbb{F}$ and $v \in \mathbb{F}$.

If $n = 1$ in $\mathbb{F}$, then (8) reduces to

$$s = v^2 - \tfrac{1}{2}.$$

For $v = \pm\frac{1}{2}$, $s = -\frac{1}{4}$ we get two points, which extend σ equidistantly to a system of $n+1$ points. If $n \neq -2$ in $\mathbb{F}$, this leads to $\omega(\mathbb{F}^n) = n+1$ according to (4). Notice that in this case $\sqrt{(n+1)/2} = 1 \in \mathbb{F}$, which supports part C(1) of Theorem 1. If simultaneously $n = 1$ and $n = -2$ in $\mathbb{F}$, then $\operatorname{char}(\mathbb{F}) = 3$. In this case we can find three points for an equidistant extension of σ:

$$y^{(1)} = y(\tfrac{1}{2}, 1), \qquad y^{(2)} = y(\tfrac{1}{2}, -1), \qquad y^{(3)} = y(-\tfrac{1}{2}, 0).$$

According to (5) no further extension is possible. So in this case $\omega(\mathbb{F}^n) = n+2$, which again corresponds to part C(1) of Theorem 1.

Suppose now $n \neq 1$ in $\mathbb{F}$. We solve (8) for s:

$$s = \frac{1}{n-1}\left(1 \pm \sqrt{1 - 2(n-1)(v^2 - \tfrac{1}{2})}\right). \tag{9}$$

There is an equidistant extension of σ if and only if the equation

$$1 - 2(n-1)(v^2 - \tfrac{1}{2}) = u^2 \quad \text{or, equivalently,}$$

$$u^2 \;+\; 2(n-1)v^2 \;=\; n \tag{10}$$

has a solution with $u \in \mathbb{F}$ and $v \in \mathbb{F}$. We try to find a solution of (10) for $v = \pm\frac{1}{2}$.

$$u^2 \;+\; 2(n-1)\tfrac{1}{4} \;=\; n, \qquad u = \pm\sqrt{(n+1)/2}.$$

If $\sqrt{(n+1)/2} \in \mathbb{F}$, then we determine $s \in \mathbb{F}$ for $v = \pm\frac{1}{2}$ by (9). The points $y^{(1)} = y(s, \frac{1}{2})$, $y^{(2)} = y(s, -\frac{1}{2})$ extend σ equidistantly to a system of $n+1$ points. If $n \neq -2$ in $\mathbb{F}$, then by (4) we conclude $\omega(\mathbb{F}^n) = n+1$. If $n = -2$ in $\mathbb{F}$, then we take a closer look at the common element s in $y^{(1)}$ and $y^{(2)}$, which we get from (9):

$$s = -\tfrac{1}{3}(1 + \sqrt{-\tfrac{1}{2}}) = -\tfrac{1}{3}(1 + \tfrac{1}{2}\sqrt{-2}).$$

Notice that $\operatorname{char}(\mathbb{F}) \neq 3$ and $\sqrt{-2} \in \mathbb{F}$ in this case. Now (10) has a further solution in $\mathbb{F}$: $v = 0,\ u = \sqrt{-2}$. For $v = 0$ we determine s_0 by (9):

$$s_0 = -\tfrac{1}{3}(1 - \sqrt{-2}).$$

It is easily checked that σ can be extended equidistantly by the following three points:

$$y^{(1)} = y(s, \tfrac{1}{2}),\ y^{(2)} = y(s, -\tfrac{1}{2}), \qquad y^{(3)} \;=\; y(s_0, 0).$$

This shows $\omega(\mathbb{F}^n) = n+2$ in this case and completes the proof of part C(1) in Theorem 1.

To finish the proof of part C(2) of Theorem 1 we may assume that n is odd and $\sqrt{(n+1)/2} \notin \mathbb{F}$. In this case the possible values of $\omega(\mathbb{F}^n)$ are $n-1$ and n. We show that it is no loss of generality, if we start with the special points studied above. For the points $x^{(1)}, \ldots, x^{(n-1)}$ of σ we set

$$w^{(j)} = x^{(j+1)} - x^{(1)}, \qquad j = 1, \ldots, n-2.$$

Then we have $(w^{(j)})^2 = 1$ and $w^{(j)} w^{(k)} = \frac{1}{2}$ for $j \neq k$. If there is any system of n points in $\mathbb{F}^n$ mutually at quadratic distance 1, then we could analogously find vectors $z^{(1)}, \ldots, z^{(n-1)}$ satisfying $(z^{(j)})^2 = 1$ and $z^{(j)} z^{(k)} = \frac{1}{2}$ for $j \neq k$. Now a theorem from linear algebra (Proposition 2 of the Appendix) guarantees the existence of an isometry f of $\mathbb{F}^n$ with

$$f(z^{(j)}) = w^{(j)} \qquad \text{for} \quad j = 1, \ldots, n-2.$$

For this proposition we need char$(\mathbb{F}) \neq 2$ and the assumption that the standard inner product of $\mathbb{F}^n$ is nonisotropic. Now

$$y = x^{(1)} + f(z^{(n-1)})$$

would have quadratic distance 1 to every point $x^{(j)},\ j < n$. This means that $\omega(\mathbb{F}^n) = n$ if and only if the system σ of points $x^{(1)}, \ldots, x^{(n-1)}$ has an equidistant extension by an additional point y. We have already shown that such an extension exists if and only if (10) is solvable in $\mathbb{F}$. This completes the proof of Theorem 1. □

3. Proof of Theorem 2

Parts A and B(1) immediately follow from Theorem 1. Let n be odd and let $(n+1)/2$ not be the square of an integer. From Theorem 1 part C(2) we conclude $\omega(\mathbb{Q}^n) = n$, if the equation

$$u^2 + 2(n-1)v^2 = n$$

has a solution with $u,\ v \in \mathbb{Q}$ and $\omega(\mathbb{Q}^n) = n-1$, if this equation is unsolvable in $\mathbb{Q}$. We may set

$$u = \frac{z}{x}, \qquad v = \frac{y}{x} \qquad \text{with integers} \quad x, y, z, x \neq 0,$$

and so arrive at the diophantine equation of Chilakamarri [4]:

$$nx^2 - 2(n-1)y^2 = z^2.$$

If this equation has a solution in integers x, y, z with $x \neq 0$, then $\omega(\mathbb{Q}^n) = n$, otherwise $\omega(\mathbb{Q}^n) = n-1$. So the proof of Theorem 2 will be accomplished, if the following proposition is shown to be true.

Proposition 1. *Let n be an integer $\neq 0$ and let $n = n_1 n_2^2$ be the factorization of n with unique squarefree divisor n_1. The diophantine equation*

$$nx^2 - 2(n-1)y^2 = z^2 \tag{11}$$

has a solution in integers $x, y, z,\ x \neq 0$, *if and only if* n_1 *has no prime divisor* $p \equiv \pm 3$ *modulo* 8.

To prove Proposition 1 we take advantage of the following result from number theory.

Lemma 1. *Let* n *be an integer* $\neq 0$ *and let* $n = n_1 n_2^2$ *be the factorization of* n *with unique squarefree divisor* n_1. *The diophantine equation*

$$x^2 - 2y^2 = n$$

has a solution in integers $x, y, z,\ x \neq 0$, *if and only if* n_1 *has no prime divisor* $p \equiv \pm 3$ *modulo* 8.

The proof of Lemma 1 is analogous to the well-known classification of those integers that can be written as the sum of two squares. A full proof of Lemma 1 can be found as Theorem 35.3 in [2].

Proof of Proposition 1. Equation (11) can be transformed to

$$n(x^2 - 2y^2) = z^2 - 2y^2. \tag{12}$$

Suppose $x, y, z, x \neq 0$, are integers satisfying (11). As $n \neq 0,\ x \neq 0$ and as $\sqrt{2}$ is irrational, the integers $x^2 - 2y^2,\ z^2 - 2y^2$ are nonzero and according to Lemma 1 have prime divisors $p \equiv \pm 3$ modulo 8 only with even multiplicity ≥ 0. Equation (12) implies that the same must be true for n.

Now let n have prime divisors $p \equiv \pm 3$ modulo 8 only with even multiplicity ≥ 0. By Lemma 1 we can find integers a, b with $a^2 - 2b^2 = n$. We may assume that $a \neq 1$. It is easily checked that

$$x = a - 1, \qquad y = b, \qquad z = a(a-1) - 2b^2$$

solves (11). □

4. Problems and Remarks

The above investigation can no doubt be extended in various directions.

1. Find a complete evaluation of $\omega(\mathbb{F}^n)$ for special fields such as finite fields or $\mathbb{Q}[\sqrt{p}]$, p a prime number.
2. Try to extend Corollary 2 to finite extensions of $\mathbb{Q}$.
3. If $\mathbb{F}$ is not a real field, then another distance function may be more appropriate.

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Appendix

To make this paper more self-contained we outline the result from linear algebra used for the proof of the last case in Theorem 1.

Assume char($\mathbb{F}$) $\neq 2$ and let the standard inner product of $\mathbb{F}^n$ be nonisotropic. An *isometry* f of $\mathbb{F}^n$ is a length preserving endomorphism of $\mathbb{F}^n$.

$$f(x)\cdot f(x) = x\cdot x \quad \text{for every} \quad x \in \mathbb{F}^n \quad \Longleftrightarrow \quad f(x)\cdot f(y) = x\cdot y \quad \text{for all} \quad x, y \in \mathbb{F}^n.$$

For $\nu \in \mathbb{F}^n$, $\nu \neq 0$ let $H(\nu) = \{x \in \mathbb{F}^n : \nu \cdot x = 0\}$ be the *hyperplane* with *normal* vector ν. Every $x \in \mathbb{F}^n$ can be uniquely written as

$$x = x_H + \lambda\nu, \qquad x_H \in H(\nu), \qquad \lambda \in \mathbb{F}.$$

The *reflection* S_ν at the hyperplane $H(\nu)$ is defined by $S_\nu(x) = x_H - \lambda\nu$. Clearly, S_ν is an isometry.

Proposition 2. *Assume that the standard inner product of* $\mathbb{F}^n$ *is nonisotropic and* char($\mathbb{F}$) $\neq 2$. *Let* $a_1, \ldots, a_m, b_1, \ldots, b_m$ *be vectors in* $\mathbb{F}^n$ *satisfying*

$$a_i \cdot a_j = b_i \cdot b_j \qquad \textit{for all} \quad i, j = 1, \ldots, m.$$

Then there is an isometry f *of* $\mathbb{F}^n$ *with* $f(a_i) = b_i$ *for every* $i = 1, \ldots, m$. *Either* $f = \mathrm{id}$ *or* f *is the product of at most* m *hyperplane reflections*.

Proof. We may start the induction on m formally with $m = 0$ and $f = \mathrm{id}$. For the inductive step suppose that $m \geq 1$ and that the assertion is true for $m - 1$. If $a_1, \ldots, a_m, b_1, \ldots, b_m$ satisfy the conditions of the proposition, then there is an isometry φ with $\varphi(a_i) = b_i$ for $i = 1, \ldots, m - 1$, where $\varphi = \mathrm{id}$ or φ is the product of at most $m - 1$ hyperplane reflections. If $\varphi(a_m) = b_m$, then we may take $f = \varphi$. Let $\nu = \varphi(a_m) - b_m \neq 0$, $f = S_\nu\varphi$. For $i < m$ we have

$$\nu \cdot \varphi(a_i) = \varphi(a_m) \cdot \varphi(a_i) - b_m \cdot \varphi(a_i) = a_m \cdot a_i - b_m \cdot b_i = 0.$$

Therefore $\varphi(a_i) \in H(\nu)$ and

$$f(a_i) = S_\nu(\varphi(a_i)) = \varphi(a_i) = b_i \qquad \text{for every} \quad i,\ 1 \leq i < m.$$

Now $(\varphi(a_m))^2 - b_m^2 = a_m^2 - b_m^2 = 0$ implies $(\varphi(a_m) - b_m) \cdot (\varphi(a_m) + b_m) = 0$. From $\nu = \varphi(a_m) - b_m$ and $(\varphi(a_m) + b_m) \in H(\nu)$ we conclude

$$\left.\begin{aligned} S_\nu(\varphi(a_m) + b_m) &= \varphi(a_m) + b_m \\ S_\nu(\varphi(a_m) - b_m) &= -\varphi(a_m) + b_m \end{aligned}\right\} \quad \Longrightarrow \quad S_\nu(\varphi(a_m)) = f(a_m) = b_m. \qquad \square$$

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