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An Alternative Proof on Four-Dimensional Zero-Sums

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Abstract. For a finite abelian group G let $\mathsf{s}(G)$ denote the smallest integer l such that every sequence S over G of length $|S| \geq l$ has a zero-sum subsequence of length $\exp(G)$. By the Erdös-Ginzburg-Ziv theorem $\mathsf{s}(C_n) = 2n - 1$. By Reiher's theorem $\mathsf{s}(C_n^2) = 4n - 3$. Elsholtz proved $\mathsf{s}(C_n^3) \geq 9n - 8$ for odd n, and Edel, Elsholtz, Geroldinger, Kubertin and Rackham have shown that for odd n: $\mathsf{s}(C_n^4) \geq 20n - 19$. In this note we give an alternative proof of this last statement.

Keywords. Zero-sums.

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Dedicated to Prof. R. Balasubramanian on the occasion of his 60th birthday.

1. Introduction

In this note we study a well known zero-sum problem, which goes back to Harborth [17]. In recent years, Adhikari, Balasubramanian, Bhowmik, the author, Gao, Geroldinger, Reiher, Schlage-Puchta, Schmid, Thangadurai, and their students, see for example [4,5,7,10,11,18], have made progress on this and related problems.

For general survey articles of the wider subject area of zero sums and its applications the reader may consult [1,7,14,16], which give many more references to the extensive literature.

For a finite abelian group G let $\mathsf{s}(G)$ denote the smallest integer l such that every sequence S over G of length $|S| \geq l$ has a zero-sum subsequence of length $\exp(G)$.

By the Erdös-Ginzburg-Ziv theorem (see [9]) $\mathsf{s}(C_n) = 2n - 1$. Reiher [19] proved that $\mathsf{s}(C_n^2) = 4n-3$. Elsholtz [8] proved $\mathsf{s}(C_n^3) \geq 9n-8$ for odd n, and Edel, Elsholtz, Geroldinger, Kubertin and Rackham [7] have shown that for odd n: $\mathsf{s}(C_n^4) \geq 20n-19$.

During recent years there have been indications that possibly even $s(C_n^3) = 9n - 8$ for all odd n might hold. In fact, this has been conjectured by Gao and Thangadurai (see [13], [12]). Moreover it has been proved for odd integers of the form $n = 3^a 5^b$ by Gao, Hou, Schmid and Thangadurai [11]. Bhowmik and Schlage-Puchta have announced an upper bound of the type $s(C_n^3) \leq (9 + o(1))n$, as n tends to infinity.

For even n, the situation is different. Here it is conjectured (see [13], [12]) that $s(C_n^3) = 8n - 7$ holds.

This is known for $n = 2^a$ and $n = 2^a 3$, see [17] and [11].

In dimension 5, Rackham (unpublished) and Edel [6] independently constructed an example showing that (for odd n): $\mathsf{s}(C_n^5) \geq 42n - 41$. Moreover, Edel lifts his construction to higher dimensions:

$$\mathsf{s}(C_n^r) \geq c_r(n-1) + 1 \text{ where } c_5 = 42, c_6 = 96, c_7 = 196.$$

In view of the above mentioned partial results in dimension 3, one might wonder about further progress in dimension 4. Quite possibly, for odd n the following holds: $\mathsf{s}(C_n^4) = 20(n-1) + 1$. Moreover, with regard to the inverse question one might expect that the extremal cases come from examples consisting of sequences consisting of $n-1$ copies of 20 distinct vectors.

A better understanding of the four-dimensional case seems desirable. For this reason we give in this note an alternative proof of the four-dimensional lower bound, and also sketch a still different variant due to Edel.

Theorem (Edel, Elsholtz, Geroldinger, Kubertin, Rackham [7]). *For odd n:* $\mathsf{s}(C_n^4) \geq 20n - 19$.

The proof given in [7] constructs 20 distinct vectors (see below) with the property that a zero-sum (modulo n) of n elements (repetition allowed) must be trivial, i.e. the zero-sum contains n times the same vector. In order to find these 20 vectors, the authors first construct an example of four vectors in dimension 2, (Lemma 3.3), with the zero-sum property in question. Then (in Lemma 3.4) the example in dimension 3 is constructed by using two versions of the two dimensional example, and adding a ninth vector.

As in one of the components one uses four times a constant entry for the first four vectors, then four times another constant entry for the second set of four vectors, it is clear that a zero-sum either restricts to the first set of four vectors, the second set of four vectors (both is impossible in view of the two dimensional property), or must use the ninth vector. As the ninth vector is well chosen, this leads to a contradiction.

Then the four-dimensional example consists of two versions of the three dimensional example and two further vectors, giving twenty vectors, (see proof of Theorem 1.1. in [7]). Again, a zero-sum cannot exist in the first or second sets of nine vectors, but the last two vectors must be used. Allowing only $n-1$ copies of the 20 vectors eventually also leads to a contradiction.

By this construction it is more or less transparent that the set of 20 vectors has the required zero-sum property, namely that a zero-sum of n elements (repetition allowed) must be trivial, i.e. n times the same vector. The proof, as just explained, is built up from individual pieces, and contains some conceptual insight.

The alternative proof we study below is more of an elementary-algebraic-algorithmic nature (similar to the three dimensional case in [8]). This approach seems more accessible to a fully automatized version in the sense of linear programming.

This approch may perhaps prove useful if one finds examples not by a general construction, as described above, but if a computer programme has tested an example, say modulo $n = 5$, and one studies whether the example works modulo general odd n.

In a final remark there is a different set to start with, which would also lead to a proof.

2. Proof of Theorem

We first study the same set of 20 vectors studied in [7].

Let $G = C_n^4$ with $n \geq 3$ odd. Let us study the multiset, consisting of $n-1$ copies of each of the following 20 vectors $\vec{v}_1, \ldots, \vec{v}_{20}$:

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\1\\0\\2\end{pmatrix} \begin{pmatrix}1\\1\\2\\0\end{pmatrix} \begin{pmatrix}1\\1\\2\\2\end{pmatrix} \begin{pmatrix}1\\2\\0\\1\end{pmatrix} \begin{pmatrix}1\\2\\1\\0\end{pmatrix} \begin{pmatrix}1\\2\\2\\1\end{pmatrix} \begin{pmatrix}1\\2\\1\\2\end{pmatrix} \begin{pmatrix}1\\3\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}2\\2\\0\\0\end{pmatrix} \begin{pmatrix}2\\2\\0\\2\end{pmatrix} \begin{pmatrix}2\\2\\2\\0\end{pmatrix} \begin{pmatrix}2\\2\\2\\2\end{pmatrix} \begin{pmatrix}2\\1\\0\\1\end{pmatrix} \begin{pmatrix}2\\1\\1\\0\end{pmatrix} \begin{pmatrix}2\\1\\2\\1\end{pmatrix} \begin{pmatrix}2\\1\\1\\2\end{pmatrix} \begin{pmatrix}2\\0\\1\\1\end{pmatrix} \begin{pmatrix}0\\2\\1\\1\end{pmatrix} \begin{pmatrix}3\\1\\1\\1\end{pmatrix}$$

Observe that the sum of coefficients of each of the vectors is even. (This will be exploited at the very end of the proof). Suppose that there is a non-trivial zero-sum. Then there are integer coefficients $0 \leq a_i \leq n-1$ so that $\sum_{i=1}^{20} a_i = n$ and $\sum_{i=1}^{20} a_i \vec{v}_i \equiv \vec{0} \bmod n$. This gives one equation and four congruences, in twenty variables (a_i). It might appear to be hopeless to solve this system of congruences in any meaningful way, but the fact that we have restricted integer variables, and that the coefficients have some structure helps us to solve the system.

Let us rewrite this in the following system of linear congruences, in some compact notation.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
(1)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$=$	n
(2)	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	0	3	$\equiv$	$0 \bmod n$
(3)	1	1	1	1	2	2	2	2	3	2	2	2	2	1	1	1	1	0	2	1	$\equiv$	$0 \bmod n$
(4)	0	0	2	2	0	1	2	1	1	0	0	2	2	0	1	2	1	1	1	1	$\equiv$	$0 \bmod n$
(5)	0	2	0	2	1	0	1	2	1	0	2	0	2	1	0	1	2	1	1	1	$\equiv$	$0 \bmod n$

The row (1) means

$$a_1 + a_2 + \cdots + a_{20} = n,$$

and the row (2) stands for

$$a_1 + a_2 + \cdots + a_9 + 2a_{10} + \cdots + 2a_{18} + 0a_{19} + 3a_{20} \equiv 0 \bmod n.$$

We want to specify the congruences, i.e. we study if these congruences can be written as an equality, with $= 0, = n, = 2n$ or similar. A notation like $(2) = n$ means that

$$a_1 + a_2 + \cdots + a_9 + 2a_{10} + \cdots + 2a_{18} + 0a_{19} + 3a_{20} = n.$$

Since all coefficients in row (5) are bounded by 2, $(5) \geq 3n$, is clearly impossible (it contradicts (1)).

We show in case 1 that $(5) = 2n$ is impossible, and in case 2 that $(5) = 0$ is impossible. It then follows (case 3) that $(5) = n$ and similarly that $(4) = n$. With some further case studies we arrive at a contradiction to the assumptions,

(1) the a_i are integers,
(2) $0 \leq a_i \leq n-1$,
(3) $\sum_{i=1}^{20} a_i = n$ and
(4) n is odd.

Case 1. Let us assume that $(5) = 2n$: Studying the linear combination 2(1)–(5) gives:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$2(1)-(5)$	2	0	2	0	1	2	1	0	1	2	0	2	0	1	2	1	0	1	1	1	$=$	0

As all coefficients on the left hand side are nonnegative, and as the right hand side is zero, this implies that

$$a_1 = a_3 = a_5 = a_6 = a_7 = a_9 = a_{10} = a_{12} = a_{14} = a_{15} = a_{16}$$
$$= a_{18} = a_{19} = a_{20} = 0.$$

Of course, this is a great simplification and actually guides us which linear combinations (such as the above 2(1)–(5)) one should look for. So our system of congruences simplifies to

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(1')$		1		1				1			1		1				1				$=$	n
$(2')$		1		1				1			2		2				2				$\equiv$	$0 \bmod n$
$(3')$		1		1				2			2		2				1				$\equiv$	$0 \bmod n$
$(4')$		0		2				1			0		2				1				$\equiv$	$0 \bmod n$
$(5')$		2		2				2			2		2				2				$=$	$2n$

(Empty boxes correspond to coefficients 0.) Row $(5')$ can be discarded as it is equivalent to row $(1')$.

In order to show that this is impossible, we distinguish two cases:

Case 1.1. If $(2') = 2n$, then considering $(2'') = 2(1') - (2')$ gives:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(2'')$		1		1				1													$=$	0

and so $a_2 = a_4 = a_8 = 0$. Now considering $(2'') = (2') - (1')$ gives

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(1'')$											1		1				1				$=$	n
$(2'')$											1		1				1				$=$	n
$(3'')$											2		2				1				$\equiv$	$0 \bmod n$
$(4'')$											0		2				1				$\equiv$	$0 \bmod n$

As not all coefficients are zero, (see also $(1'')$), we have that $(3'') > 0$. As the coefficients in $(3'')$ are bounded by 2: $(3'') \leq 2n$.

If $(3'') = n$, then $2(2'') - (3'')$ implies that $a_{17} = n$, in contradiction to $a_i < n$.

If $(3'') = 2n$, then $(3'') - 2(2'')$ gives $a_{17} = 0$. Row $(1'')$, i.e. $a_{11} + a_{13} = n$ and $0 \leq a_i < n$ implies that $a_{13} > 0$. But then row $(4'')$ gives a contradiction with $2a_{13} \equiv 0 \bmod n$, n odd and $0 < a_{13} < n$.

Case 1.2. This case is very similar to the one above. If row $(2') = n$, then considering $(2'') = (2') - (1')$ gives:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(2'')$											1		1				1				$=$	0

and so $a_{11} = a_{13} = a_{17} = 0$. Then $(2'') = 2(1') - (2')$ gives

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(1'')$		1		1				1													$=$	n
$(2'')$		1		1				1													$=$	n
$(3'')$		1		1				2													$\equiv$	$0 \bmod n$
$(4'')$		0		2				1													$\equiv$	$0 \bmod n$

This system is almost equivalent (but $(3'')$ differs), to the one in case 1.1. As before we show this leads to a contradiction. Note that $0 \leq (3'') \leq 2n$. If $(3'') = n$, then $(3'') - (2'')$ shows that $a_8 = 0$. But then by $(1'')$ and $a_2 < n$ one necessarily has $a_4 > 0$. Equation $(4'')$ means $2a_4 \equiv 0 \bmod n$, which is impossible since $0 < a_4 < n$ and n is odd. If $(3'') = 2n$, then $(3'') - (2'')$ implies $a_8 = n$, which is impossible since $a_8 < n$. This completes the proof of the first case.

Case 2. Assume that $(5) = 0$. This implies that

$$\begin{aligned} a_2 = a_4 = a_5 = a_7 = a_8 = a_9 = a_{11} = a_{13} = a_{14} = a_{16} \\ = a_{17} = a_{18} = a_{19} = a_{20} = 0. \end{aligned}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(1')$	1		1			1				1		1			1						$=$	n
$(2')$	1		1			1				2		2			2						$\equiv$	$0 \bmod n$
$(3')$	1		1			2				2		2			1						$\equiv$	$0 \bmod n$
$(4')$	0		2			1				0		2			1						$\equiv$	$0 \bmod n$
$(5')$	0		0			0				0		0			0						$=$	0

This situation is equivalent to that solved in Case 1: Equation $(5')$ can be discarded in both systems. Moreover, a_1 corresponds to a_2, a_3 to a_4, a_6 to a_8, a_{10} to a_{11}, a_{12} to a_{13}, and a_{15} to a_{17}. Hence this situation also leads to a contradiction.

Case 3. The only case that remains is $(5) = n$. Now, looking at the initial system of linear congruences, we observe that there is a symmetry between line (4) and (5): One can exchange vectors (or alternatively exchange columns) $\vec{v}_2$ with $\vec{v}_3$, $\vec{v}_5$ with $\vec{v}_6$, $\vec{v}_7$ with $\vec{v}_8$, $\vec{v}_{11}$ with $\vec{v}_{12}$, $\vec{v}_{14}$ with $\vec{v}_{15}$, $\vec{v}_{16}$ with $\vec{v}_{17}$.

Just as before one can assume $(4) = 2n$ or $(4) = 0$. Both cases are discarded, by exactly the same argument as before. Therefore it also follows that $(4) = n$. Now let us study rows (2) and (3). Since the coefficients 0 and 3 occur only once in row (2): $(2) \geq 3n$ would imply that $a_{20} = n$, and $(2) = 0$ would imply that $a_{19} = n$; both of these cases contradict $a_i < n$. Similarly $(3) \geq 3n$ and $(3) = 0$ is impossible. So we have $(2) = n$ or $(2) = 2n$, and the same for row (3).

Case 3.1. First we show that (2) $=$ (3) $= 2n$ is impossible. Suppose that (2) $=$ (3) $= 2n$ holds.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
4(1)-(2)-(3)	2	2	2	2	1	1	1	1	0	0	0	0	0	1	1	1	1	2	2	0	$=$	0

This implies that

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_{14} = a_{15} = a_{16} = a_{17} = a_{18} = a_{19} = 0.$$

The system of congruences then simplifies to

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(1')$									1	1	1	1	1							1	$=$	n
$(2')$									1	2	2	2	2							3	$=$	$2n$
$(3')$									3	2	2	2	2							1	$=$	$2n$
$(4')$									1	0	0	2	2							1	$=$	n
$(5')$									1	0	2	0	2							1	$=$	n

Now $(2') - 2(1')$ gives $-a_9 + a_{20} = 0$, i.e. $a_9 = a_{20}$. But then $(5')$ becomes $2a_9 + 2a_{11} + 2a_{13} = n$ which is a contradiction, since n is odd.

Case 3.2. Now assume that (2) $=$ (3) $= n$ holds.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$-2(1) + (2) + (3)$	0	0	0	0	1	1	1	1	2	2	2	2	2	1	1	1	1	0	0	2	$=$	0

This implies that

$$a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = a_{13} = a_{14} = a_{15}$$
$$= a_{16} = a_{17} = a_{20} = 0.$$

The system of congruences then simplifies to

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
$(1')$	1	1	1	1														1	1		$=$	n
$(2')$	1	1	1	1														2	0		$=$	n
$(3')$	1	1	1	1														0	2		$=$	n
$(4')$	0	0	2	2														1	1		$=$	n
$(5')$	0	2	0	2														1	1		$=$	n

$(2') - (3')$ shows that $a_{18} = a_{19}$. This contradicts $(5')$, since n is odd.

Case 3.3. Now let us finally assume that one of the rows (2) or (3) is $= n$ and the other one is $= 2n$. (The argument below is symmetric so that it does not matter which one is $= n$).

We now invoke the observation made above, that the sum of coefficients of each of the 20 vectors is even. Hence we study the sum $t = (2) + (3) + (4) + (5)$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20		
(2) + (3) +(4) + (5)	2	4	4	6	4	4	6	6	6	4	6	6	8	4	4	6	6	4	4	6	$=$	$5n$

As all coefficients on the left hand side above are even, by this property, the sum t must be an even number. On the right hand side we have $5n$, which is a contradiction, since n is odd.

This completes the proof of the Theorem.

3. Another alternative proof

Yves Edel has found the following set of 20 vectors, which also only allows trivial zero-sums of n elements.

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}2\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}2\\0\\1\\0\end{pmatrix} \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \begin{pmatrix}2\\0\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\1\\1\end{pmatrix} \begin{pmatrix}1\\2\\1\\1\end{pmatrix} \begin{pmatrix}1\\1\\2\\1\end{pmatrix} \begin{pmatrix}0\\2\\2\\1\end{pmatrix} \begin{pmatrix}2\\2\\2\\1\end{pmatrix} \begin{pmatrix}1\\1\\1\\2\end{pmatrix} \begin{pmatrix}0\\2\\1\\2\end{pmatrix} \begin{pmatrix}2\\2\\1\\2\end{pmatrix} \begin{pmatrix}0\\1\\2\\2\end{pmatrix} \begin{pmatrix}2\\1\\2\\2\end{pmatrix} \begin{pmatrix}1\\2\\2\\2\end{pmatrix}$$

Observe that here a) only entries $0, 1, 2$ occur, b) that the sum of coefficients for each vector is odd, and c) that the four unit vectors occur. A variation of the explicit proof above would work as well. Note that in this example the frequencies of integers are not according the $20 = 9 + 9 + 2$ scheme, which was explicitly used in the proof of [7] and implicitly in the proof above. The frequencies of entries are of type $20 = 6+8+6$: Each of the four coordinates contains six times a 0, eight times a 1 and six times a 2. (Note that also in the proof above, in view of several well chosen linear combinations, we could typically reduce the system with 20 variables, to a system with 6 variables.)

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