04. Base, Subbase, Neighbourhood base

The collection of all open sets in a topological space may be quite complicated or difficult to define. Therefore it is desirable to have simpler "building blocks" for the topology.

Definition. Let (X, τ) be a topological space.

A subcollection $\mathcal{B} \subseteq \tau$ is called a **base** of (X, τ) if

$$\forall\, O \in \tau \;\; \forall\, x \in O \quad \exists\, B_x \in \mathcal{B} \quad \text{such that} \quad x \in B_x \subseteq O$$

or, equivalently, if every nonempty open set is the union of certain members of $\mathcal{B}$.

Remark. Of course, τ itself is a base for (X, τ) .

Remark. Every base $\mathcal{B}$ for (X, τ) has a certain cardinality $|\mathcal{B}|$.

$$w(X) = \min\{|\mathcal{B}| \;:\; \mathcal{B} \text{ is a base for } (X, \tau)\}$$

is called the **weight** of (X, τ) .

If $w(X) \leq \aleph_0$, i.e. there exists a countable base, then (X, τ) is called **second countable** or an A_2-**space**.

Examples.

1) If (X, d) is a metric space with topology τ_d , then

$$\mathcal{B} = \{K(x, \varepsilon) \;:\; x \in X \;,\; \varepsilon > 0\}$$

is a base for (X, τ_d) .

2) Consider $\mathbb{R}$ (resp. $\mathbb{R}^n$) with the usual metric and topology. Then

$$\mathcal{B} = \{(q - \tfrac{1}{n}, q + \tfrac{1}{n}) \;:\; q \in \mathbb{Q} \;,\; n \in \mathbb{N}\}$$

(resp. $\mathcal{B} = \{K(q, \frac{1}{n}) \;:\; q \in \mathbb{Q}^n \;,\; n \in \mathbb{N}\}$)

is a countable base, therefore each $\mathbb{R}^n$ (with the usual topology) is second countable.

We observe that $\aleph_0 = w(X) < |\mathbb{R}^n| = c$.

3) Let τ be the discrete topology on X .

Every base $\mathcal{B}$ for (X, τ) must contain the sets $\{x\}$, $x \in X$.

This family $\{\{x\} \ , \ x \in X\}$ is obviously a base, therefore $w(X) = |X|$.

Proposition. Let $w(X) \leq \alpha$ and let $\{O_i \ : \ i \in I\}$ be a family of open sets in (X, τ) .

Then there is a subset $I_0 \subseteq I$ such that $|I_0| \leq \alpha$ such that

$$\bigcup_{i \in I_0} O_i = \bigcup_{i \in I} O_i \ .$$

Proof. Clearly $\bigcup_{i \in I_0} O_i \subseteq \bigcup_{i \in I} O_i$ for each $I_0 \subseteq I$.

Now let $\mathcal{B}$ be a base with $|\mathcal{B}| \leq \alpha$.

Let $\mathcal{B}_0 = \{B \in \mathcal{B} \ : \ \exists \ i \in I$ such that $B \subseteq O_i\}$ and for each $B \in \mathcal{B}_0$ choose $i_B \in I$ such that $B \subseteq O_{i_B}$.

If $I_0 = \{i_B \ : \ B \in \mathcal{B}_0\}$ then $|I_0| \leq \alpha$.

Now let $x \in \bigcup_{i \in I} O_i$. Then there exists $j \in I$ such that $x \in O_j$ and $B \in \mathcal{B}$ such that $x \in B \subseteq O_j$.

Then $B \in \mathcal{B}_0$, $i_B \in I_0$ and $x \in B \subseteq O_{i_B} \subseteq \bigcup_{i \in I_0} O_i$.

Therefore $\bigcup_{i \in I_0} O_i = \bigcup_{i \in I} O_i$. $\square$

Theorem. Let $\aleph_0 \leq w(X) \leq \alpha$ and $\mathcal{B}$ be a base for (X, τ) .

Then there exists a base $\mathcal{B}_0$ such that $\mathcal{B}_0 \subseteq \mathcal{B}$ and $|\mathcal{B}_0| \leq \alpha$.

Proof. Let $\mathcal{B} = \{B_i \ : \ i \in I\}$.

Choose a base $\mathcal{B}_1 = \{W_j \ : \ j \in J\}$ with $|J| \leq \alpha$.

For each $j \in J$ let $I_j = \{i \in I \ : \ B_i \subseteq W_j\}$, i.e. $W_j = \bigcup_{i \in I_j} B_i$.

According to the previous proposition there exists $I_j^* \subseteq I_j$ such that $|I_j^*| \leq \alpha$ and $W_j = \bigcup_{i \in I_j^*} B_i$.

Let $\mathcal{B}_0 = \{B_i \ : \ i \in I_j^* \ , \ j \in J\}$. Then $|\mathcal{B}_0| \leq \alpha$.

We claim that $\mathcal{B}_0$ is a basis. Let $O \subseteq X$ be open and $x \in O$. Then there exists $j \in J$ such that $x \in W_j \subseteq O$.

Consequently there is $i \in I_j^*$ such that $x \in B_i \subseteq W_j \subseteq O$. Since $B_i \in \mathcal{B}_0$ we are done. $\square$.

Every base $\mathcal{B}$ for a space (X, τ) obviously has the following properties.

(B1) $\forall \ x \in X \ \ \exists \ B \in \mathcal{B} \ \ : \ \ x \in B$

(B2) $\forall \ B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ such that

$x \in B_3 \subseteq B_1 \cap B_2$.

((B1) holds because X is open, and (B2) holds because $B_1 \cap B_2$ is open.)

These properties can be utilized for the **construction** of topologies on a given set.

Let X be a **set** (!) and let $\mathcal{B}$ be a family of subsets of X that satisfies (B1) and (B2).

Then there is a unique topology τ on X such that $\mathcal{B}$ is a base for (X, τ) .

(This resembles the construction of the topology of a metric space.)

Let $\tau = \{\emptyset\} \cup \{O \subseteq X \ : \ \forall \ x \in O \ \ \exists \ B_x \in \mathcal{B}$ such that $x \in B_x \subseteq O\}$.

We first show that τ is a topology on X . Clearly, $\emptyset \in \tau$ and $X \in \tau$ (since (B1)).

Let $O_1, O_2 \in \tau$ and $x \in O_1 \cap O_2$.

Then there exist $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \subseteq O_1$ and $x \in B_2 \subseteq O_2$.

By (B2) , there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq O_1 \cap O_2$.

Therefore $O_1 \cap O_2 \in \tau$.

Let $O_i \in \tau$, $i \in I$ and let $O = \bigcup_{i \in I} O_i$.

For each $x \in O$ there exists $j \in I$ such that $x \in O_j$ and $B \in \mathcal{B}$ such that $x \in B \subseteq O_j \subseteq O$. Thus $O \in \tau$.

According to the construction of τ , $\mathcal{B}$ is obviously a base for (X, τ) .

Let σ be another topology on X for which $\mathcal{B}$ is a base.

If $O \in \sigma$ then O is the union of sets of $\mathcal{B}$, therefore $O \in \tau$ and thus $\sigma \subseteq \tau$. In the same manner one shows that $\tau \subseteq \sigma$, therefore $\sigma = \tau$.

Example. (Sorgenfrey line)

Let $X = \mathbb{R}$ and consider $\mathcal{B} = \{[a, b) \ : \ a, b \in \mathbb{R} \text{ and } a < b\}$.

It is easy to see that (B1) and (B2) are satisfied therefore exists a unique topology τ for which $\mathcal{B}$ is a base.

This space is called the **Sorgenfrey line**.

Some of the properties of the Sorgenfrey line are:

1) Each half-open interval $[a, b)$ is open and closed in (X, τ) .

2) $\tau_d \subseteq \tau$ but $\tau_d \neq \tau$.

3) $\overline{\mathbb{Q}} = \mathbb{R}$

4) (X, τ) is **not** second countable.

(Proofs as exercise)

Definition. Let (X, τ) be a topological space.

A subcollection $\mathcal{S} \subseteq \tau$ is called a **subbase** of (X, τ) , if the family of finite intersections of members of $\mathcal{S}$ is a base of (X, τ) .

This means, whenever $O \in \tau$ and $x \in O$ there exist $S_1, S_2, \ldots, S_k \in \mathcal{S}$ such that $x \in S_1 \cap S_2 \cap \ldots \cap S_k \subseteq O$.

Remarks.

1) Every base of (X, τ) is also a subbase.

2) Let $X = \mathbb{R}$ have the usual topology.

Then $\mathcal{S} = \{(-\infty, b) \ : \ b \in \mathbb{R}\} \cup \{(a, \infty) \ : \ a \in \mathbb{R}\}$ is a subbase but not a base.

Now let (X, τ) be a space and $\mathcal{S}$ be a subbase. Then

(SB) $\forall\, x \in X \ \ \exists\, S \in \mathcal{S}$ such that $x \in S$

(SB) holds because X is open. So a subbase is, in particular, a so-called **covering** of X .

The property (SB) can again be utilized for the construction of topological spaces.

Let X be a set and $\mathcal{S}$ be a family of subsets of X satisfying (SB) .

Then there is a unique topology τ on X such that $\mathcal{S}$ is a subbase of (X, τ) .

Proof. Let $\mathcal{B}$ be the family of finite intersections of members of $\mathcal{S}$. In particular, $\mathcal{S} \subseteq \mathcal{B}$.

If $x \in X$ there exists $S \in \mathcal{S}$ such that $x \in S$, therefore (B1) holds.

Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$.

Then there exist $S_1, \ldots, S_n, S_{n+1}, \ldots, S_{n+m} \in \mathcal{S}$ such that

$$B_1 = S_1 \cap \ldots \cap S_n \quad \text{and} \quad B_2 = S_{n+1} \cap \ldots \cap S_{n+m}$$

Since $B_3 = S_1 \cap \ldots S_{n+m} = B_1 \cap B_2$ and $x \in B_3$, (B2) holds.

Hence there is a unique topology τ on X having $\mathcal{B}$ as a base, and obviously $\mathcal{S}$ as a subbase. $\Box$

Remark. τ is the coarsest topology on X where all sets of $\mathcal{S}$ are open. We also say that τ is "generated" by $\mathcal{S}$.

Proof. Let σ be a topology on X with $\mathcal{S} \subseteq \sigma$. Then $\mathcal{B} \subseteq \sigma$ and consequently $\tau \subseteq \sigma$. $\square$

Examples.

1) Let $X = \mathbb{R}$ and $\mathcal{S}$ be the family of **all** halfopen Intervals.

Then $\mathcal{S}$ generates the discrete topology τ .

For $x \in \mathbb{R}$ we have $[x, x+1) \cap (x-1, x] = \{x\} \in \tau$, i.e. each singleton is open.

For $A \subseteq \mathbb{R}$, $A = \bigcup\limits_{x \in A} \{x\} \in \tau$.

2) Let $(X, <)$ be a linearly ordered set $(|X| > 1)$.

For each $x \in X$ let

$$(\leftarrow, x) = \{y \in X \ : \ y < x\} \ \text{ and } \ (x, \rightarrow) = \{y \in X \ : \ y > x\} \ .$$

Then $\mathcal{S} = \{(\leftarrow, x) \ , \ (x, \rightarrow) \ : \ x \in X\}$ generates the so-called **order topology** on $(X, <)$.

Remarks.

(a) The order topology on $\mathbb{R}$ (with the usual order) is the usual topology (generated by the metric). The order topology on $\mathbb{N}$ (with the usual order) is the discrete topology.

(b) Typical open neighbourhoods (depending on the order) can be intervals of the form (a, b) , $[x, a)$, $(a, x]$.

3) The **product topology** (see later).

4) The **weak topology** with respect to a family of functions (see later).

Let X be a set and $\{f_i : X \to \mathbb{R} \ : \ i \in I\}$ be a family of functions.

The weak topology on X with respect to the given family of functions is the coarsest topology on X which makes all f_i continuous.

Obviously, $\mathcal{S} = \{f_i^{-1}(O) \ : \ O$ open in $\mathbb{R}$, $i \in I\}$ is a subbase for this topology.

Definition. Let (X, τ) be a space and let $x \in X$.

A family $\mathcal{B}(x) \subseteq \mathcal{U}(x)$ of neighbourhoods of x is called a **neighbourhood base** in x if

$$\forall\, U \in \mathcal{U}(x) \ \ \exists\, B \in \mathcal{B}(x) \ \text{ such that } \ B \subseteq U \ .$$

If all members of $\mathcal{B}(x)$ are open (resp. closed) we speak of an **open neighbourhood base** (resp. closed neighbourhood base).

Example. For a metric space (X, d) and $x \in X$ is

$$\mathcal{B}(x) = \{K(x, \tfrac{1}{n}) \ : \ n \in \mathbb{N}\}$$

a countable (!) open neighbourhood base in x .

Definition. (X, τ) is called **first countable** or A_1-**space** if each point has a countable open neighbourhood base.

Remarks. Let $X = \mathbb{R}$.

(a) If τ_d is the usual topology then (X, τ_d) is first countable.

(b) If σ is the topology of the Sorgenfrey line then (X, σ) is first countable.

(c) If ρ is the cofinite topology then (X, ρ) is **not** first countable.

(d) If τ is the discrete topology then (X, τ) is first countable but **not** second countable.

Remark. Let $\mathcal{B}(x) = (B_n)_{n \in \mathbb{N}}$ be a countable neighbourhood base in $x \in X$. Then there exists a **nested** neighbourhood base in x .

Let $U_n = B_1 \cap \ldots \cap B_n$ for each $n \in \mathbb{N}$. Since the finite intersection of neighbourhoods is a neighbourhood, $(U_n)_{n\in\mathbb{N}}$ is clearly also a neighbourhood base satisfying $U_{n+1} \subseteq U_n \subseteq B_n$ for each $n \in \mathbb{N}$.

Now let (X, τ) be a space and for each $x \in X$ let $\mathcal{B}(x)$ be an **open** neighbourhood base in x .

Then the following properties obviously hold

(UB 1) $\mathcal{B}(x) \neq \emptyset \quad \forall\, x \in X$ and $x \in B \quad \forall\, B \in \mathcal{B}(x)$

(UB 2) $B_1, B_2 \in \mathcal{B}(x) \;\Rightarrow\; \exists\, B_3 \in \mathcal{B}(x)$ such that $B_3 \subseteq B_1 \cap B_2$

(UB 3) $y \in B$ and $B \in \mathcal{B}(x) \;\Rightarrow\; \exists\, B^* \in \mathcal{B}(y)$ such that $B^* \subseteq B$.

Again these properties can be utilized for the construction of topologies.

Let X be a set. For each $x \in X$ let $\mathcal{B}(x)$ be a family of sets such that (UB 1) - (UB3) are satisfied.

Then there is a unique topology τ on X such that for each x , $\mathcal{B}(x)$ is an open neighbourhood base in x .

Proof.

Let $\tau = \{\emptyset\} \cup \{O \subseteq X \;:\; \forall\, x \in O \;\exists\, B_x \in \mathcal{B}(x)$ such that $B_x \subseteq O\}$.

Then (TR 1) and (TR 3) obviously hold.

Let $O_1, O_2 \in \tau$ and let $x \in O_1 \cap O_2$. Then there exist $B_1, B_2 \in \mathcal{B}(x)$ such that $x \in B_1 \subseteq O_1$ and $x \in B_2 \subseteq O_2$.

(UB 2) $\Rightarrow\; \exists\, B_3 \in \mathcal{B}(x)$ such that $B_3 \subseteq B_1 \cap B_2 \subseteq O_1 \cap O_2$.

Therefore $O_1 \cap O_2 \in \tau$ and (TR2) holds and τ is a topology on X .

(UB 3) $\Rightarrow\; B \in \tau \;\; \forall\, B \in \mathcal{B}(x)$

According to the definition of τ , each family $\mathcal{B}(x)$ is an open neighbourhood base in x .

It is easy to see that τ is uniquely determined. $\square$

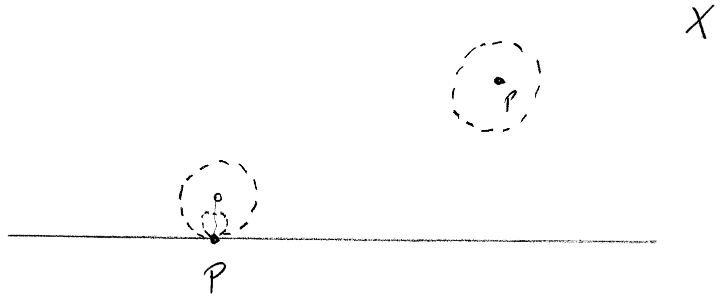
Example. (Niemitzky plane)

Let $X = \{(x,y) \in \mathbb{R}^2 \ : \ y \geq 0\}$ be the upper half plane in $\mathbb{R}^2$.

Let $p = (x,y) \in X$.

If $y > 0$ then $\mathcal{B}(p)$ consists of all open discs with center p and radius $\frac{1}{n}$ that do **not** intersect the $x-$axis.

If $y = 0$ then $\mathcal{B}(p)$ consists of all sets which are the union of $\{p\}$ and an open disc with center $(x, \frac{1}{n})$ and radius $\frac{1}{n}$.



Obviously, (UB 1) - (UB 3) are satisfied.

The resulting space is called the **Niemitzky plane**. Clearly it is first countable but **not** second countable (Exercise!).