05. Continuous functions

Definition. Let (X, τ) , (Y, σ) be topological spaces , $f : X \to Y$ a function and $x_0 \in X$.

1) f is called **continuous at** $x_0 \in X$ if

   $\forall\, V \in \mathcal{U}(f(x_0))\ \exists\, U \in \mathcal{U}(x_0)$ such that $f(U) \subseteq V$.

2) f is called (globally) **continuous** if f is continuous at each $x \in X$.

Remark. If (X, d) and (Y, ρ) are metric spaces with topologies τ_d and σ_ρ then a function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon)$ such that

$$d(x_0, x) < \delta \ \Rightarrow\ \rho(f(x_0), f(x)) < \varepsilon\ .$$

Theorem. Let (X, τ) , (Y, σ) be spaces and $f : X \to Y$. Then the following are equivalent:

1) f is continuous

2) $\forall\, V \in \sigma\ :\ f^{-1}(V) \in \tau$ (i.e. preimages of open sets are open)

3) If $\mathcal{S}$ is a subbase of (Y, σ) then $f^{-1}(S) \in \tau$ for each $S \in \mathcal{S}$

4) $B \subseteq Y$ closed in $(Y, \sigma)\ \Rightarrow\ f^{-1}(B) \subseteq X$ is closed in (X, τ)

5) $\forall\, A \subseteq X\ :\ f(\overline{A}) \subseteq \overline{f(A)}$

6) $\forall\, B \subseteq Y\ :\ \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

Proof.

$1) \Rightarrow 2)$: If $V \in \sigma$ and $x \in f^{-1}(V)$ then $f(x) \in V$ and therefore $V \in \mathcal{U}(f(x))$.

By assumption, there exists $U_x \in \mathcal{U}(x)$ with $f(U_x) \subseteq V$.

Then $U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) \in \mathcal{U}(x)$. Therefore $f^{-1}(V) \in \tau$.

$2) \Rightarrow 3)$: Is trivial.

$3) \Rightarrow 2)$: Let $\mathcal{S}$ be a subbase of (Y, σ) such that $f^{-1}(S) \in \tau$ for each $S \in \mathcal{S}$.

If $V \in \sigma$ and $x \in f^{-1}(V)$ then $f(x) \in V$.

Then there exist $S_1, S_2, \ldots, S_k \in \mathcal{S}$ such that

$f(x) \in S_1 \cap S_2 \cap \ldots \cap S_k \subseteq V \quad \Rightarrow \quad x \in f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_k) \subseteq f^{-1}(V)$

By assumption, $f^{-1}(S_1) \cap \ldots \cap f^{-1}(S_k) \in \tau$.

Therefore $f^{-1}(V) \in \mathcal{U}(x)$ and consequently $f^{-1}(V) \in \tau$.

$2) \Rightarrow 4)$: Let $B \subseteq Y$ be closed. Then $Y \setminus B \in \sigma$ and

$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \in \tau$. Thus $f^{-1}(B) \subseteq X$ is closed.

$4) \Rightarrow 5)$: Let $A \subseteq X$. If $B = \overline{f(A)}$ then B is closed and $f(A) \subseteq B$.

By assumption, $f^{-1}(B)$ is closed and $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(B)$.

Hence $\overline{A} \subseteq f^{-1}(B)$ and $f(\overline{A}) \subseteq B = \overline{f(A)}$.

$5) \Rightarrow 6)$: Let $B \subseteq Y$ and let $A = f^{-1}(B)$.

By assumption, $f(\overline{A}) \subseteq \overline{f(A)}$, i.e. $f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}$.

Thus $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

$6) \Rightarrow 1)$: Let $x_0 \in V$, $V \in \mathcal{U}(f(x_0))$ and let $B = Y \setminus V$.

Since $f(x_0) \in \text{int}V$ we have $f(x_0) \notin Y \setminus \text{int}V = \overline{Y \setminus V} = \overline{B}$.

So $x_0 \notin f^{-1}(\overline{B})$ and, by assumption, $x_0 \notin \overline{f^{-1}(B)}$.

Hence there exists $U \in \mathcal{U}(x)$ such that $U \cap f^{-1}(B) = \emptyset$.

Consequently $\emptyset = f(U) \cap B = f(U) \cap (Y \setminus V)$ and $f(U) \subseteq V$.

Thus f is continuous at an arbitrary point $x_0 \in X$. $\Box$

Corollary. Let X, Y, Z be spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous.

Then $g \circ f : X \to Z$ is continuous.

Proof. Let $W \subseteq Z$ be open in Z. Then $g^{-1}(W) \subseteq Y$ is open in Y and $f^{-1}(g^{-1}(W) = (g \circ f)^{-1}(W) \subseteq X$ is open in X. $\square$

In a similar manner one can show that if f is continuous at $x_0 \in X$ and g is continuous at $y_0 = f(x_0) \in Y$ then $g \circ f$ is continuous at x_0.

Examples.

1) If τ is the discrete topology on X.

Then **every** function $f : (X, \tau) \to (Y, \sigma)$ is continuous.

2) Let τ, σ be topologies on X.

Then $\tau \subseteq \sigma$ if and only if the identity function $id : (X, \sigma) \to (X, \tau)$ is continuous.

3) **Constant** functions are always continuous.

Let $f : (X, \tau) \to (Y, \sigma)$ with $f(x) = y_0 \quad \forall\, x \in X$.

If $V \in \sigma$ then $f^{-1}(V) = X$ if $y_0 \in V$ and $f^{-1}(V) = \emptyset$ if $y_0 \notin V$.

4) Let (X, τ) be a space and let have $\mathbb{R}$ the usual topology.

Consider $C(X) = \{f : X \to \mathbb{R} \ : \ f \text{ is continuous}\}$.

One can show that for $f, g \in C(X)$ and $\lambda \in \mathbb{R}$ that

$f + g$, $f - g$, fg , $\lambda f \in C(X)$

$\frac{f}{g} \in C(X)$ whenever $g(x) \neq 0$ for all $x \in X$

$|f|$, $\min\{f, g\}$, $\max\{f, g\} \in C(X)$

$C(X)$ is called the **ring of continuous functions** on X.

5) (Exercise) Characterize the continuous functions $f : (X, \tau) \to \mathbb{R}$ where τ is the cofinite topology on X.

Definition. Let (X, τ) be a space.

A sequence (f_n) of functions $f_n : X \to \mathbb{R}$ **converges uniformly** to a function $f : X \to \mathbb{R}$ if

$$\forall\, \varepsilon > 0 \;\; \exists\, N \in \mathbb{N} \text{ such that } |f(x) - f_n(x)| < \varepsilon \quad \forall\, x \in X \;,\; \forall\, n \geq N$$

Theorem. If the $f_n : X \to \mathbb{R}$ are continuous and converge uniformly to $f : X \to \mathbb{R}$ then f is continuous.

Proof. Let $x_0 \in X$ and $\varepsilon > 0$.

Then $\exists\, N \in \mathbb{N}$ with $|f(x) - f_n(x)| < \frac{\varepsilon}{3} \quad \forall\, x \in X \;,\; \forall\, n \geq N$.

Since f_N is continuous there exists $U \in \mathcal{U}(x_0)$ such that

$$|f_N(x_0) - f_N(x)| < \tfrac{\varepsilon}{3} \quad \forall\, x \in U$$

For $x \in U$ we now have

$|f(x_0) - f(x)| = |(f(x_0) - f_N(x_0)) + (f_N(x_0) - f_N(x)) + (f_N(x) - f(x))| \leq$

$|f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| <$

$\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

Therefore f is continuous at $x_0 \in X$. $\square$