06. Initial and final topology

We consider the following problem:

Given a set (!) X and a family (Y_i, σ_i) of spaces and corresponding functions $f_i : X \to Y_i$, $i \in I$.

Find a topology τ on X such that all functions $f_i : (X, \tau) \to (Y_i, \sigma_i)$ become continuous.

It is obvious that the discrete topology on X fulfills the requirement. Therefore we look for the possibly coarsest topology on X that fulfills the requirement.

Since continuity means that the inverse images of open sets are open, we consider the family

$$\mathcal{S} = \{f_i^{-1}(V_i) \ : \ V_i \text{ open in } Y_i \ , \ i \in I\}$$

From a previous discussion we know that there is a unique topology τ on X having $\mathcal{S}$ as a subbase, and that it is the coarsest topology making all sets of $\mathcal{S}$ open.

Definition. τ is called the **initial topology** on X with respect to the functions $f : X \to Y_i$, $i \in I$.

Remarks.

1) Let X be a set and (Y, σ) a space and $f : X \to Y$.

Then the initial topology τ on X with respect to f is

$$\tau = \{f^{-1}(V) \ : \ V \in \sigma\} .$$

2) Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. Then

$$X' = \{f : X \to \mathbb{K} \ : \ f \text{ is linear and bounded}\}$$

is called the dual space of X .

The initial topology on X with respect to all $f \in X'$ is called the **weak**

topology on X (it is, in general, coarser than the topology induced by the norm and not metrizable).

The initial topology has the following "universal property":

Let τ be the initial topology on X with respect to the family of functions $\{f_i : X \to (Y_i, \sigma)\}$, $i \in I\}$.

Let (Z, ρ) be a space and $g : Z \to X$ a function. Then

g is continuous $\Leftrightarrow$ $f_i \circ g : Z \to Y_i$ is continuous $\forall\, i \in I$

Proof.

If g is continuous then $f_i \circ g$ is continuous for each $i \in I$ since each f_i is continuous.

Conversely, $\{f_i^{-1}(V_i) \ : \ V_i$ open in Y_i , $i \in I\}$ is a subbase of (X, τ) .

By assumption, each $g^{-1}(f_i^{-1}(V_i)) = (f_i \circ g)^{-1}(V_i)$ is open in Z .

Therefore, by a previous theorem, g is continuous. $\square$

The subspace topology

Let (X, τ) be a space and $A \subseteq X$.

The initial topology on A with respect to the inclusion function $j : A \to X$ where $j(x) = x \ \ \forall\, x \in A$, is called the **subspace topology** on A and denoted by $\tau|_A$.

Obviously, $\tau|_A = \{j^{-1}(O) = O \cap A \ : \ O \in \tau\}$.

$(A, \tau|_A)$ is called a **subspace** of (X, τ) , and $G \subseteq A$ is called **open in** A if $G \in \tau|_A$.

(So the sets open in A can be represented as an intersection of an open set in X and A .)

Example. Let $X = \mathbb{R}$ with the usual topology and $A = [0, 2)$.

Then $[0, 1) = (-1, 1) \cap A$ is open in A but not in X .

The proof of the following result is left as an exercise.

Proposition. Let (X, τ) be a space and $A \subseteq X$.

1) $B \subseteq A$ is closed in A $\Leftrightarrow$

there exists $F \subseteq X$ closed in X such that $B = F \cap A$.

2) For $B \subseteq A$, the closure of B with respect to $(A, \tau|_A)$ is denoted by $\overline{B}^A$.

Then $\overline{B}^A = \overline{B} \cap A$.

However, we only have $\text{int}B \cap A \subseteq \text{int}_A B$ in general.

($\text{int}_A B$ is the interior of B with respect to $(A, \tau|_A)$) .

3) Let (X, d) be a metric space and $A \subseteq X$.

Then A itself is a metric space by the induced metric $d|_{A \times A}$.

It holds that the topology on A generated by $d|_{A \times A}$ coincides with the subspace topology $\tau_d|_A$.

4) Let $B \subseteq A \subseteq X$.

If B is open (resp. closed) in X then B is open (resp. closed) in A .

If A is open in X and B is open in A then B is open in X .

If A is closed in X and B is closed in A then B is closed in X .

Definition. $A \subseteq X$ is called a **discrete subspace** of (X, τ) if $\tau|_A$ is the discrete topology on A .

Exercise. Show that $\mathbb{R}$ with the usual topology has a countable discrete subspace but **not** an uncountable discrete subspace.

Show that the Niemitzky plane has an uncountable discrete subspace.

The product topology

For each $i \in I$ let (X_i, τ_i) be a space.

The **product set** $X = \prod_{i \in I} X_i$ is (by definition) the set of all functions $x : I \to \bigcup_{i \in I} X_i$ such that $x(j) \in X_j \ \forall\, j \in I$.

(An element of the product set is obtained by "choosing" an element from each set X_i .)

We use the notation $x = (x_i)_{i \in I}$ or $x = (x_i)$ where $x_i = x(i)$.

x_i is called the ith **coordinate** (or **component**) of x .

If I is finite, say $I = \{1, 2, \ldots, n\}$ we write $X = X_1 \times X_2 \times \ldots \times X_n$ and $x = (x_1, x_2, \ldots, x_n)$.

For each $i \in I$ there exists a canonical function, the **ith projection**

$$p_i : X = \prod_{i \in I} X_i \to X_i \quad \text{where } \ p_i(x) = x_i \ .$$

Note that each p_i is surjective.

Definition. The initial topology on $X = \prod_{i \in I} X_i$ with respect to the family $\{p_i \ : \ i \in I\}$ is called the **product topology** τ on X .

Remark. According to previous results

1) Each p_i is continuous.

2) A function $f : Y \to \prod_{i \in I} X_i$ is continuous if and only if the "component functions" $p_i \circ f : Y \to X_i$ are continuous for each $i \in I$.

The function $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ with $f(t) = (\cos t, t^2)$ is continuous because the functions $f_1(t) = \cos t$ and $f_2(t) = t^2$ are continuous.

Definition. A function $f : (X, \tau) \to (Y, \sigma)$ between any spaces is an **open** function (resp. a **closed** function) if

$\forall\, O \in \tau \ : \ f(O) \in \sigma$

(resp. $\forall\, A$ closed in (X, τ) : $f(A)$ is closed in (Y, σ))

Remark. (Proof as exercise)

Let $\mathcal{B}$ be a base for (X,τ) . Then $f:(X,\tau)\to(Y,\sigma)$ is open if and only if $f(B)\in\sigma \quad \forall\, B\in\mathcal{B}$.

Let $X=\prod_{i\in I} X_i$ have the product topology τ .

Then $\mathcal{S}=\{p_i^{-1}(O_i) \ : \ i\in I$ and $O_i\subseteq X_i$ open in $X_i\}$ is subbase for (X,τ) .

Note that $p_i^{-1}(O_i)=\{x\in X \ : \ x_i\in O_i\}$.

A typical member of the resulting base for (X,τ) has the form

$$B=p_{i_1}^{-1}(O_{i_1})\cap\ldots\cap p_{i_k}^{-1}(O_{i_k})=\{x\in X \ : \ x_{i_1}\in O_{i_1},\ldots,x_{i_k}\in O_{i_k}\}$$

where $i_1,\ldots,i_k\in I$ and $O_{i_j}\subseteq X_{i_j}$ open in X_{i_j} .

(In the finite case $X=X_1\times\ldots\times X_n$ we have

$$B=p_1^{-1}(O_1)\cap\ldots\cap p_n^{-1}(O_n)=O_1\times\ldots\times O_n \ )$$

Now let $B=p_{i_1}^{-1}(O_{i_1})\cap\ldots\cap p_{i_k}^{-1}(O_{i_k})$ and $i\in I$.

If $i=i_j\in\{i_1,\ldots,i_k\}$ then $p_i(B)\subseteq O_i$. If $x_i\in O_i$ it is possible to "construct" an element $x\in B$ having x_i as its ith coordinate.

Therefore $p_i(B)=O_i$.

If $i\notin\{i_1,\ldots,i_k\}$ then, taking any $x_i\in X_i$, it is possible to "construct" an element $x\in B$ having x_i as its ith coordinate.

Therefore $p_i(B)=X_i$. Now we have

Proposition. Each projection $p_i:X\to X_i$ is an open function.

Remark. If $X_i=Y$ for each $i\in I$, we write for the product space

$$X=Y^I=\{x:I\to Y\} \ .$$

Hence the set of all functions $x:\mathbb{R}\to\mathbb{R}$ can be written as the product set $\mathbb{R}^{\mathbb{R}}$.

The resulting product topology is called the **topology of pointwise convergence**.

Examples.

1) Consider $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$.

One the one hand we have the topology generated by the metric on $\mathbb{R}^n$, and on the other hand we have the product topology (where $\mathbb{R}$ has the usual topology).

Those two topologies coincide!

(Hint for the proof in the case $n = 2$: each open ball contains an open square and conversely)

2) For each $i \in I$ let τ_i be the discrete topology on X_i .

Then the product topology τ is discrete if and only if I is finite.

Proof. Let I be finite, i.e. $X = X_1 \times \ldots \times X_n$.

Then $\{x\} = \{x_1\} \times \ldots \times \{x_n\}$ is open in X for $x = (x_1, \ldots, x_n)$.

Conversely, suppose that I is infinite and assume that $\{x\} \in \tau$ for $x \in X$.

Then there exist $i_1, \ldots, i_k \in I$ and corresponding open sets $O_{i_1}, \ldots, O_{i_k}$ such that

$$\{x\} = p_{i_1}^{-1}(O_{i_1}) \cap \ldots \cap p_{i_k}^{-1}(O_{i_k})$$

It is possible to choose $j \notin \{i_1, \ldots, i_k\}$ and $y_j \in X_j$ with $y_j \neq x_j$.

"Construct" $y \in X$ with the y_j and $y_i = x_i$ for $i \neq j$.

Then $y \neq x$ but $y \in p_{i_1}^{-1}(O_{i_1}) \cap \ldots \cap p_{i_k}^{-1}(O_{i_k})$, a contradiction. $\square$

3) For each $n \in \mathbb{N}$ let $X_n = \{0, 1\}$ have the discrete topology.

Then $X = \prod_{n \in \mathbb{N}} X_n = \{0, 1\}^{\mathbb{N}}$ consists of all sequences containing only 0 or 1 .

This space is called the **Cantor cube** .

We now consider a related problem than the previous one.

Given a set (!) Y and a family (X_i, τ_i) of spaces and corresponding functions $f_i : X_i \to Y$, $i \in I$.

Find a topology σ on Y such that all functions $f_i : (X_i, \tau_i) \to (Y, \sigma)$ become continuous.

It is obvious that the indiscrete topology on Y fulfills the requirement. Therefore we look for the possibly finest topology on Y that fulfills the requirement.

It is easily checked that

$$\sigma = \{V \subseteq Y \ : \ f_i^{-1}(V) \in \tau_i \quad \forall\, i \in I\}$$

is, in fact, a topology, and also the finest topology on Y such that all $f_i : X_i \to Y$ are continuous.

σ is called the **final topology** on Y with respect to the functions $f_i : X_i \to Y$, $i \in I$.

Remark. Also the final topology has a "universal property".

If $g : Y \to Z$ is a mapping then g is continuous if and only $g \circ f_i : X_i \to Z$ is continuous for each $i \in I$.

Proof. If g is continuous then clearly all functions $g \circ f_i$, $i \in I$ are continuous.

Conversely, suppose that all $g \circ f_i$ are continuous. Let $W \subseteq Z$ be open in Z and let $V = g^{-1}(W) \subseteq Y$.

Since $f_i^{-1}(V) = f_i^{-1}(g^{-1}(W)) = (g \circ f_i)^{-1}(W) \in \tau_i$ for each $i \in I$ it follows that $V \in \sigma$ and that g is continuous. $\square$

The sum topology

Let (X_i, τ_i) , $i \in I$ be a family of spaces such that $X_{i_1} \cap X_{i_2} = \emptyset$ whenever $i_1 \neq i_2$.

Consider $X = \bigcup_{i \in I} X_i$.

For each $i \in I$ we have the canonical inklusion $j_i : X_i \to X$ where $j_i(x) = x$ for each $x \in X_i$.

The final topology τ on X with respect to $\{j_i : i \in I\}$ is called the **sum topology** .

We also write $X = \bigoplus_{i \in I} X_i$ or $X = \sum_{i \in I} X_i$.

Let $i^* \in I$. Then for each $i \in I$ we have $j_i^{-1}(X_{i^*}) = X_{i^*}$ whenever $i = i^*$, and $j_i^{-1}(X_{i^*}) = \emptyset$ whenever $i \neq i^*$.

Consequently, each subset $X_i \subseteq X$ is open and closed in (X, τ) .

Furthermore, a subset $V \subseteq X$ is open in (X, τ) if and only if $V \cap X_i \in \tau_i \quad \forall \, i \in I$.

Remark. Sometimes it is desirable to consider sums of the same spaces, for example $\mathbb{R} \oplus \mathbb{R}$.

In such a case we can construct the sum of the spaces $X_1 = \mathbb{R} \times \{1\}$ and $X_2 = \mathbb{R} \times \{2\}$.

The quotient topology

Let (X, τ) be a space and " $\sim$ " be an equivalence relation X .

Then the set of all (different) equivalence classses is denoted by

$$X/_\sim = \{[x] \ : \ x \in X\} \quad \text{where} \quad [x] = \{y \in X \ : \ y \sim x\} \ .$$

We have also a canonical surjective function $\pi : X \to X/_\sim$ defined by $\pi(x) = [x]$.

The final topology σ on $X/_\sim$ with respect to the function π is called the **quotient topology**.

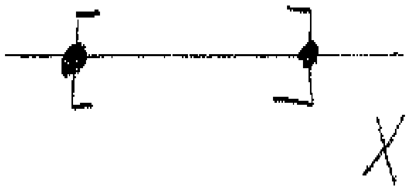
$(X/_\sim, \sigma)$ is called **quotient space**.

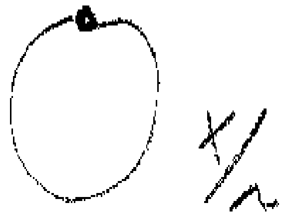
Clearly, $\sigma = \{W \subseteq X/_\sim \ : \ \pi^{-1}(W) \in \tau\}$.

Examples.

1) Let $X = [0,1] \subseteq \mathbb{R}$ have the usual topology.

Only 0 and 1 are equivalent to each other and, of course, each point is equivalent to itself.





We obtain, at least "geometrically", $X/_\sim \simeq S^1$ where

$$S^n = \{x \in \mathbb{R}^{n+1} \ : \ \sum_{i=1}^{n+1} x_i^2 = 1\}$$

is the $n-$**dimensional sphere** .

2) Let $X = \mathbb{R}^2$ have the usual topology and

$$(x_1, y_1) \sim (x_2, y_2) \quad \Leftrightarrow \quad x_1 - x_2 \in \mathbb{Z} \text{ and } y_1 - y_2 \in \mathbb{Z}$$

An equivalence class is a lattice of points in the plane. One can show that

$$\mathbb{R}^2/_\sim \simeq S^1 \times S^1$$

for which a geometrical interpretation is the 2-dimensional **Torus**.

3) Let $X = [0,1] \times [0,1]$ be the unit square.

If we "identify" **all** points of the boundary then $X/_\sim$ can be interpreted geometrically as the surface of the sphere S^2 .

In a similar way one can obtain the surface of a cylinder and the Moebius strip.

Equivalence relations can be obtained by functions.

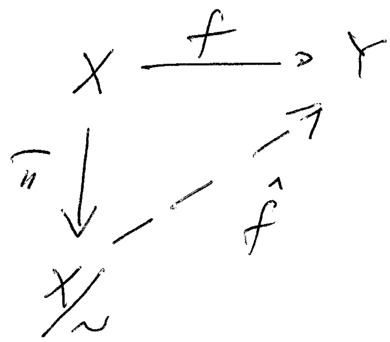
Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then there is a natural equivalence relation on X , namely

$x_1 \sim x_2 \quad \Leftrightarrow \quad f(x_1) = f(x_2)$

We observe also that the function

$\widehat{f} : X/_\sim \to Y \quad , \quad \widehat{f}([x]) = f(x)$

is well-defined (!), therefore the diagram



is commutative, i.e. $f = \widehat{f} \circ \pi$.

By a previous result, f is continuous if and only if $\widehat{f}$ is continuous. Furthermore, f is surjective if and only if $\widehat{f}$ is surjective.

Next we observe that $\widehat{f}$ is injective:

$\widehat{f}([x_1]) = \widehat{f}([x_2]) \;\Rightarrow\; f(x_1) = f(x_2) \;\Rightarrow\; x_1 \sim x_2 \;\Rightarrow\; [x_1] = [x_2]$

Hence: If f is surjective and continuous, then $\widehat{f}$ is bijective and continuous.

However, the inverse function $(\widehat{f})^{-1} : Y \to X/_\sim$ need not be continuous in general (i.e. Y and $X/_\sim$ need not be homeomorphic).

Proposition. If f is surjective, continuous and, in addition, an open function (or a closed function) then $(\widehat{f})^{-1}$ is continuous.

Proof. (For the case that f is an open function)

First observe that $\widehat{f}(W) = f(\pi^{-1}(W))$ for $W \subseteq X/_\sim$.

If $W \subseteq X/_\sim$ is open then $\pi^{-1}(W) \subseteq X$ is open.

Since f is an open function, $f(\pi^{-1}(W)) = \widehat{f}(W) \subseteq Y$ is open.

Therefore $\widehat{f}$ is an open function.

But this means that $(\widehat{f})^{-1} : Y \to X/_\sim$ must be continuous because $((\widehat{f})^{-1})^{-1}(W) = \widehat{f}(W)$ (i.e. the inverse image of an open set in $X/_\sim$ under the function $(\widehat{f})^{-1}$ is open in Y).

Example. Consider $f : [0,1] \to S^1 \subseteq \mathbb{R}^2$, $f(t) = (\cos 2\pi t, \sin 2\pi t)$.

Then f is surjective and continuous.

We will see later that f is a closed function.

Thus $\widehat{f}$ is bijective and continuous **and** $(\widehat{f})^{-1}$ is continuous.

Observe that $f(t_1) = f(t_2) \quad \Leftrightarrow \quad t_1 = 0, t_2 = 1$ or $t_1 = 1, t_2 = 0$.