07. Homeomorphisms and embeddings

Homeomorphisms are the isomorphisms in the category of topological spaces and continuous functions.

Definition.

1) A function $f : (X, \tau) \to (Y, \sigma)$ is called a **homeomorphism** between (X, τ) and (Y, σ) if f is bijective, continuous **and** the inverse function $f^{-1} : (Y, \sigma) \to (X, \tau)$ is continuous.

2) (X, τ) is called **homeomorphic** to (Y, σ) if there is a homeomorphism $f : X \to Y$.

Remarks.

1) Being homeomorphic is apparently an equivalence relation on the class of all topological spaces.

2) It is a fundamental task in General Topology to decide whether two spaces are homeomorphic or not.

3) Homeomorphic spaces (X, τ) and (Y, σ) cannot be distinguished with respect to their topological structure because a homeomorphism $f : X \to Y$ not only is a bijection between the elements of X and Y but also yields a bijection between the topologies via $O \mapsto f(O)$.

Hence a property whose definition relies only on set theoretic notions and the concept of an open set holds if and only if it holds in each homeomorphic space. Such properties are called **topological properties**.

For example, "first countable" is a topological property, whereas the notion of a bounded subset in $\mathbb{R}$ is not a topological property.

Example. The function $f : \mathbb{R} \to (-1, 1)$ with $f(t) = \frac{t}{1+|t|}$ is a homeomorphism, the inverse function is $f^{-1}(x) = \frac{x}{1-|x|}$.

The function $g : (-1,1) \to (a,b)$ with $g(x) = \frac{b-a}{2}x + \frac{a+b}{2}$ is a homeomorphism.

Therefore all open intervals in $\mathbb{R}$ are homeomorphic to each other and homeomorphic to $\mathbb{R}$.

Exercise. Show that $\mathbb{R}$ is **not** homeomorphic to an interval $[a,b)$.

Theorem. Let $f : (X,\tau) \to (Y,\sigma)$ be bijective and continuous. Then the following are equivalent:

1) f is a homeomorphism,

2) f is an open function,

3) f is a closed function.

Proof. Obviously, f is open $\Leftrightarrow$ f^{-1} is continuous $\Leftrightarrow$ f is closed.

(The inverse image of $A \subseteq X$ under f^{-1} is $(f^{-1})^{-1}(A) = f(A)$) $\square$

Definition. A function $f : (X,\tau) \to (Y,\sigma)$ is called an **embedding** if f is a homeomorphism between (X,τ) and $(f(X), \sigma|_{f(X)})$.

This means that f is injective and continuous and for each $O \in \tau$ the set $f(O)$ is open in the subspace $f(X)$.

Example.

The function $f : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (x,0)$ is an embedding.

So the $x-$axis in $\mathbb{R}^2$ is topologically "the same" as the real line $\mathbb{R}$.

Example.

The function $f : [0,1) \to \mathbb{R}^2$ with $f(t) = (\cos 2\pi t, \sin 2\pi t)$ is injective and continuous but **not** an embedding.

Observe that $f([0,1)) = S^1$. S^1 is compact (see later) whereas $[0,1)$ is not. Therefore f cannot be an embedding.