08. Separation axioms

Separation axioms provide information if there are "enough" open sets to "separate" points resp. subsets. The presence of certain separation properties has often important consequences.

Definition. A space (X, τ) is called

1) T_0-**space** if for two distinct points $x, y \in X$ there exists an open set $O \in \tau$ containing one point but not the other.

2) T_1-**space** if for two distinct points $x, y \in X$ there exist open sets $O_x, O_y \in \tau$ such that $x \in O_x$ $y \notin O_x$ and $y \in O_y$ $x \notin O_y$.

3) T_2-**space** or **Hausdorff space** if for two distinct points $x, y \in X$ there exist open sets $O_x, O_y \in \tau$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$.

Remarks.

(a) Clearly, T_2-space $\Rightarrow$ T_1-space $\Rightarrow$ T_0-space

(b) The indiscrete topology on an infinite set is not a T_0-space.

(c) The Sierpinski Space (X, τ) where $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$ is T_0 but not T_1 .

(d) Let X be an infinite set and τ the cofinite topology on X . If $x \neq y$ then $O_x = X \setminus \{y\}$ and $O_y = X \setminus \{x\}$ are open neighbourhoods of x resp. y showing that (X, τ) is a T_1-space.

We saw earlier that any two nonempty open sets intersect therefore (X, τ) cannot be T_2 .

(e) The topology τ_d of a metric space (X, d) is T_2 . For $x \neq y$ let $r = \frac{1}{2}d(x, y) > 0$. Then $K(x, r) \cap K(y, r) = \emptyset$ by the triangle inequality.

Theorem. For (X, τ) the following are equivalent:

1) (X, τ) is a T_1-space,

2) $\bigcap\{U \ : \ U \in \mathcal{U}(x)\} = \{x\} \quad \forall \, x \in X$,

3) $\{x\}$ is closed $\forall x \in X$.

Proof. Very easy! $\square$

Hence in T_1-spaces singletons and thus finite subsets are always closed.

Theorem. For (X, τ) the following are equivalent:

1) (X, τ) is a T_2-space,

2) $\bigcap\{\overline{U} \ : \ U \in \mathcal{U}(x)\} = \{x\} \quad \forall \, x \in X$,

3) the diagonal $\Delta = \{(x, x) \ : \ x \in X\} \subseteq X \times X$ is closed in $X \times X$.

Proof.

1) $\Rightarrow$ 2): Clearly, $x \in \bigcap\{\overline{U} \ : \ U \in \mathcal{U}(x)\}$. If $y \neq x$ there are open neighbourhoods O_x, O_y of x resp. y with $O_x \cap O_y = \emptyset$.

It follows that $\overline{O_x} \cap O_y = \emptyset$ and thus $y \notin \overline{O_x}$.

2) $\Rightarrow$ 3): Let $(x, y) \notin \Delta$, i.e. $x \neq y$. By assumption, there exists a neighhourhood, and thus also an open neighbourhood $O_x \in \mathcal{U}(x)$ such that $y \notin \overline{O_x}$.

Let $O_y = X \setminus \overline{O_x}$. Then O_y is an open neighbourhood of y with $O_x \cap O_y = \emptyset$ and therefore $(O_x \times O_y) \cap \Delta = \emptyset$. So Δ is closed.

3) $\Rightarrow$ 1): Let $x \neq y$. Then $(x, y) \neq \Delta$.

By assumption there exists a member $U \times V$ of the canonical base for $X \times X$ containing (x, y) such that $(U \times V) \cap \Delta = \emptyset$.

Hence $U \cap V = \emptyset$. $\square$

Remark. We say that in a space (X, τ) a sequence (x_n) **converges to** $x \in X$, $x_n \to x$, if

$$\forall \, U \in \mathcal{U}(x) \quad \exists \, N \in \mathbb{N} \quad \text{such that} \ \ x_n \in U \quad \forall \, n \geq N \, .$$

Let X is an infinite set and τ the cofinite topology on X . If (x_n) is a sequence with pairwise distinct members then (x_n) converges to **every** $x \in X$. So the convergence of sequences is, in general, **not** unique.

If a space (X, τ) is T_2 then a sequence can obviously converge to at most one point.

Remark. Another fundamental property of T_2-spaces is the following:

Let $f, g : (X, \tau) \to (Y, \sigma)$ be continuous functions and (Y, σ) be a T_2-space.

Then the function $F : X \to Y \times Y$ with $F(x) = (f(x), g(x))$ is also continuous.

Since (Y, σ) is T_2 , the diagonal is closed in $Y \times Y$ and therefore

$A = F^{-1}(\Delta) = \{x \in X \ : \ f(x) = g(x)\}$ is closed in (X, τ) .

In particular, if f and g coincide on a dense subset $D \subseteq X$, i.e. a subset with $\overline{D} = X$, then $A = X$ (because $D \subseteq A \Rightarrow \overline{D} \subseteq A$).

So, $f|_D = g|_D \Rightarrow f = g$.

Definition. A space (X, τ) is called

1) T_3-**space** if for each closed set $A \subseteq X$ and each point $x \notin A$ there exist open sets $U, V \in \tau$ such that

$x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$

2) $T_{3a}-$**space** if for each closed set $A \subseteq X$ and each point $x \notin A$ there exists a continuous function $f : X \to [0, 1]$ such that

$f(x) = 1$ and $f(a) = 0 \quad \forall\, a \in A$

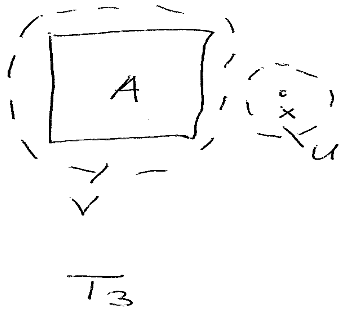
3) T_4-**space** if for any closed sets A, B with $A \cap B = \emptyset$ there exist open sets $U, V \in \tau$ such that

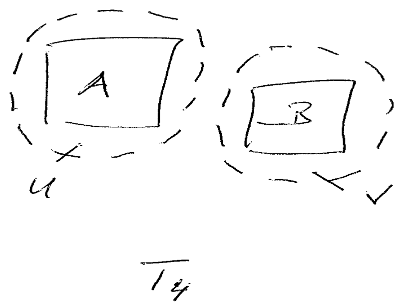
$A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$

4) **regular** if T_3-space and T_1-space

5) **completely regular** if T_{3a}−space and T_1−space

6) **normal** if T_4−space and T_1−space .





It is clear that

normal $\Rightarrow$ T_4 , completely regular $\Rightarrow$ T_{3a} , regular $\Rightarrow$ T_3 .

One can show that these implications cannot be reversed in general.

Proposition. Moreover, the following holds for a space (X, τ) :

1) regular $\Rightarrow$ T_2 ,

2) completely regular $\Rightarrow$ regular ,

3) normal $\Rightarrow$ completely regular.

Proof.

Ad 1) : If $x \neq y$ then $\{y\}$ is closed and $x \neq \{y\}$.

Ad 2) : We show that a T_{3a}−space is T_3 . So let $A \subseteq X$ be closed and $x \notin A$.

By assumption there is a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ and $f(a) = 0 \quad \forall\, a \in A$.

The intervals $(\frac{1}{2}, 1]$ and $[0, \frac{1}{2})$ are disjoint and open in $[0, 1]$, therefore $U = f^{-1}((\frac{1}{2}, 1])$ and $V = f^{-1}([0, \frac{1}{2}))$ are open sets in (X, τ) .

Clearly, $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

Ad 3) : This is a consequence of the Lemma of Urysohn, a fundamental result in General Topology (see later). $\square$

Remark. Suppose that $A, B \subseteq X$ and there is a function $f : X \to \mathbb{R}$ such that $f(a) = r \quad \forall\, a \in A$ and $f(b) = s \quad \forall\, b \in B$ where $r \neq s$.

There are disjoint open neighbourhoods $W_1, W_2 \subseteq \mathbb{R}$ of r resp. s .

If $U = f^{-1}(W_1)$ and $V = f^{-1}(W_2)$ then $U, V \subseteq X$ are disjoint open neighbourhoods of A resp. B .

Theorem. Every metric space (X, d) is normal.

Proof. We know from calculus that for each subset $\emptyset \neq A \subseteq X$ the function

$$d_A : X \to \mathbb{R} \quad \text{where} \quad d_A(x) = d(A, x) = \inf\{d(a, x) \ : \ a \in A\}$$

is continuous and $d_A(x) = 0 \ \Leftrightarrow\ x \in \overline{A}$.

Now let $A, B \subseteq X$ be closed and $A \cap B = \emptyset$. Observe that in this case $d_A(x) + d_B(x) \neq 0 \quad \forall\, x \in X$.

The function $f : X \to \mathbb{R}$ with $f(x) = \frac{d_A(x)}{d_A(x)+d_B(x)}$ is continuous with $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.

By the previous remark, A and B are therefore contained in disjoint open sets. $\square$

Corollary. A space (X, τ) that is not normal cannot be metrizable.

We now address the question about the preservation of separation axioms when forming subspaces and products.

Theorem. Let (X, τ) be a T_j-space, where $j \in \{0, 1, 2, 3, 3a\}$ and let $A \subseteq X$. Then $(A, \tau|_A)$ is a T_j-space.

Proof. For $j = 3$ (the other cases are similar).

Let $B \subseteq A$ be closed in A and let $x \in A \setminus B$.

There is a set $F \subseteq X$ closed in (X, τ) such that $B = F \cap A$.

Since $x \notin F$ there exist $O_1, O_2 \in \tau$ such that $x \in O_1$, $F \subseteq O_2$ and

$O_1 \cap O_2 = \emptyset$.

If $V_1 = O_1 \cap A$, $V_2 = O_2 \cap A$ then V_1, V_2 are open in A with

$x \in V_1$, $B = F \cap A \subseteq O_2 \cap A = V_2$ and $V_1 \cap V_2 = \emptyset$. $\square$

Remarks.

(i) In general, the result does **not** hold for T_4-spaces.

(ii) However, a **closed** subspace of a T_4-space is again a T_4-space (proof as exercise).

(iii) In a metric space, **every** subspace is normal (since every subspace is a metric space). This property is called **hereditarily normal**.

Theorem. For each $i \in I$ let (X_i, τ_i) be a T_j-space, where $j \in \{0, 1, 2, 3, 3a\}$.

Then $X = \prod_{i \in I} X_i$ with the product topology τ is a T_j-space.

Proof. For $j = 3a$ (the other cases are similar).

Let $A \subseteq X = \prod_{i \in I} X_i$ be closed and $x^* \notin A$.

Then there is a set of the canonical base $B = p_{i_1}^{-1}(O_{i_1}) \cap \ldots \cap p_{i_k}^{-1}(O_{i_k})$ such that $x^* \in B$ and $B \cap A = \emptyset$.

For each $r \in \{1, 2, \ldots, k\}$ we have $x_{i_r}^* \in O_{i_r}$ resp. $x_{i_r}^* \notin X_{i_r} \setminus O_{i_r}$.

By assumption there is a continuous function $f_{i_r} : X_{i_r} \to [0, 1]$ such that

$f_{i_r}(x_{i_r}^*) = 1$ and $f_{i_r}(y_{i_r}) = 0$ for $y_{i_r} \in X_{i_r} \setminus O_{i_r}$.

Consider $f : X \to [0, 1]$ where

$f(x) = \min\{f_{i_1} \circ p_{i_1}(x), \ldots, f_{i_k} \circ p_{i_k}(x)\} = \min\{f_{i_1}(x_{i_1}), \ldots, f_{i_k}(x_{i_k})\}$.

Then f is continuous and $f(x^*) = 1$.

Now let $y \in A$. Then $y \notin B$.

Hence there is i_r such that $y \notin p_{i_r}^{-1}(O_{i_r})$, i.e. $y_{i_r} \in X_{i_r} \setminus O_{i_r}$.

Thus $f_{i_r}(y_{i_r}) = 0$ and so $f(y) = 0$. $\square$

Remark. One can show that even the product of two normal spaces need not be T_4 .

Theorem. (**Lemma of Urysohn**)

Let (X, τ) be a T_4–space, $A, B \subseteq X$ be closed with $A \cap B = \emptyset$.

Then there is a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for each $a \in A$ and $f(b) = 1$ for each $b \in B$.

Proof.

For each $r \in \mathbb{Q} \cap [0, 1]$ we construct an open set $V_r \subseteq X$ with the following properties:

(1) $\overline{V_r} \subseteq V_s$ whenever $r < s$

(2) $A \subseteq V_0$ and $B \subseteq X \setminus V_1$

This construction is done inductively.

By hypothesis, there exist open sets O_1, O_2 with $A \subseteq O_1$, $B \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$. Hence $\overline{O_1} \cap O_2 = \emptyset$ and $\overline{O_1} \cap B = \emptyset$.

Let $V_0 = O_1$ and $V_1 = X \setminus B$.

Then V_0, V_1 are open, $A \subseteq V_1$, $B \subseteq X \setminus V_1$ and $\overline{V_0} \subseteq V_1$.

We write $(0, 1) \cap \mathbb{Q}$ as a sequence $r_3, r_4, r_5, \dots$ and we set $r_1 = 0$ and $r_2 = 1$.

Then (2) and

(3_k) $\overline{V_{r_i}} \subseteq V_{r_j}$ if $r_i < r_j$ for $i, j \leq k$

holds for $k = 2$.

Suppose the V_{r_i} have been defined for $i \leq n$ and (3_n) holds ($n \geq 2$).

Choose $r_l, r_m \in \{r_1, r_2, \dots, r_n\}$ such that r_l is the closest number to r_{n+1} from the left, and r_m is the closest number to r_{n+1} from the right.

Since $r_l < r_m$ we have $\overline{V_{r_l}} \subseteq V_{r_m}$ resp. $\overline{V_{r_l}} \cap (X \setminus V_{r_m}) = \emptyset$.

The sets $\overline{V_{r_l}}$ and $X \setminus V_{r_m}$ are disjoint and closed. Hence there exist open sets W_1, W_2 such that

$$\overline{V_{r_l}} \subseteq W_1 \ , \ (X \setminus V_{r_m}) \subseteq W_2 \quad \text{and} \quad W_1 \cap W_2 = \emptyset \ \ (\Rightarrow \ \overline{W_1} \cap W_2 = \emptyset).$$

Let $V_{r_{n+1}} = W_1$. Then $\overline{V_{r_l}} \subseteq V_{r_{n+1}} \subseteq \overline{V_{r_{n+1}}} \subseteq V_{r_m}$.

Thus (3_{n+1}) holds and the sequence $V_{r_1}, V_{r_2}, V_{r_3}, \ldots$ satisfies the conditions (1) and (2) .

Now define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{r : x \in V_r\} & \text{if } x \in V_1 \\ 1 & \text{if } x \notin V_1 \end{cases}$$

Then $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B \subseteq X \setminus V_1$.

We now show that f is continuous. Sets of the form $[0,a)$, $a \leq 1$ and $(b,1]$, $b \geq 0$ form a subbase for $[0,1]$.

Since $f(x) < a \;\Leftrightarrow\; \exists\, r < a$ such that $x \in V_r$,

we have $f^{-1}([0,a)) = \bigcup_{r<a} V_r$ which is an open set.

If $f(x) > b$ there exist r, r' with $b < r < r' < f(x)$.

Then $x \notin V_{r'}$ and $x \notin \overline{V_r}$. Therefore

$$f^{-1}((b,1]) = \bigcup_{r>b} (X \setminus \overline{V_r}) \quad \text{which is an open set.}$$

Therefore f is continuous. $\square$

Without proof we mention a very important consequence of the lemma of Urysohn which is known as the **Tietze extension theorem**.

Theorem. Let (X, τ) be a T_4- space.

Let $A \subseteq X$ be closed and $f : A \to [0,1]$ be continuous.

Then f can be continuously extended to X , i.e. there exists a continuous function $f^* : X \to [0,1]$ such that $f^*|_A = f$.

(The result remains valid if $[0,1]$ is replaced by $\mathbb{R}$.)