09. Convergence

I. Sequences

Definition.

Let (X, τ) be a space, (x_n) a sequence in X and $x \in X$.

1) (x_n) **converges** to x , $x_n \to x$, if

$\forall\ U \in \mathcal{U}(x)\ \ \exists\ N \in \mathbb{N}$ such that $x_n \in U \quad \forall\ n \geq N$.

2) x is called an **accumulation point** of (x_n) if each neighbourhood of x contains infinitely many members of (x_n) .

Remark. Let (X, τ) and (Y, σ) be spaces, $A \subseteq X$ and $x \in X$.

It is easily seen that if $a_n \to x$ where $a_n \in A\ \ \forall\ n \in \mathbb{N}$ then $x \in \overline{A}$.

Moreover, if $f : X \to Y$ is continuous in x and $x_n \to x$ then $f(x_n) \to f(x)$.

The converse, however, need not be true in general!

Example. There is an uncountable well ordered set Y such that each element of Y has at most countably many predecessors.

(Take a well ordering " $<$ " of $\mathbb{R}$. If every element has only countably many predecessors, let $Y = \mathbb{R}$.

Otherwise let z be the smallest element having uncountably many predecessors and let $Y = \{y \in \mathbb{R}\ :\ y < z\}$.)

We now add a largest element to this well ordered set. Take any $a \notin Y$ and let $X = Y \cup \{a\}$. Extend the order to X by $y < a\ \ \forall\ y \in A$.

X is again well ordered and can be written in the form $X = [0, a]$ where 0 denotes the smallest element with respect to the well ordering.

Let τ be the order topology on X .

If (y_n) is a sequence in Y then $\bigcup_{n\in N} [0, y_n]$ is countable (!) hence there is $y \in Y$ such that $y_n < y \;\; \forall\, n \in \mathbb{N}$.

$(y, a]$ is a neighbourhood of a so (y_n) does **not** converge to a .

On the other hand, we clearly have $a \in \overline{Y}$.

Consider $f : X \to \{0, 1\}$ with $f(y) = 0 \;\; \forall\, y \in Y$ and $f(a) = 1$ then f satisfies the condition "$x_n \to x \;\Rightarrow\; f(x_n) \to f(x)$" but is **not** continuous at $a \in X$.

Remark. This unsatisfactory state of affairs leads to the necessity to generalize the notion of a sequence. There are two equivalent approaches to accomplish this.

A sequence in X is a function $x : \mathbb{N} \to X$. If we replace $\mathbb{N}$ by any **directed set** $\mathbb{D}$ we obtain the notion of a **net** $x : \mathbb{D} \to X$ in X .

(A directed set is a partial ordered set where each two elements have an upper bound.)

The other generalization is via **filters** that we discuss later.

Remark. If (X, τ) is first countable (in particular, a metric space) then the closure of subsets and the continuity of functions $f : X \to Y$ can be described by sequences.

II. Filter

The notion of a filter can already be defined on a set. To define convergence of a filter, we need a topological space.

Definition. Let $X \neq \emptyset$ be a set.

1) A nonempty family $\mathcal{F}$ of subsets of X is called a **filter on** X if

(F1) $\emptyset \notin \mathcal{F}$

(F2) $F_1, F_2 \in \mathcal{F} \;\;\Rightarrow\;\; F_1 \cap F_2 \in \mathcal{F}$

(F3) $F \in \mathcal{F}$ and $F \subseteq F' \;\;\Rightarrow\;\; F' \in \mathcal{F}$

2) A nonempty family $\mathcal{B}$ of subsets of X is called a **filter base on** X if

(FB1) $\emptyset \notin \mathcal{B}$

(FB2) $B_1, B_2 \in \mathcal{B} \;\Rightarrow\; \exists\, B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Remarks.

1) Every filter is a filter base.

2) To each filter base $\mathcal{B}$ on X we can assign

$$\mathcal{F} = \{F \subseteq X \;:\; \exists\, B \in \mathcal{B} \text{ such that } B \subseteq F\}$$

$\mathcal{F}$ is in fact a filter, the filter **generated by** $\mathcal{B}$.

3) A nonempty family $\mathcal{B}$ of subsets of X with $\emptyset \notin \mathcal{B}$ which is closed under forming finite intersections is obviously a filter base.

4) Let $\mathcal{F}$ be a filter on X . Then

$$\tau = \{\emptyset\} \cup \{F \;:\; F \in \mathcal{F}\} \quad \text{is a topology on } X .$$

Examples.

1) Let X be a set and $\emptyset \neq A \subseteq X$.

Then $\mathcal{B} = \{A\}$ is a filter base.

$\mathcal{F} = \{F \subseteq X \;:\; A \subseteq F\}$ is called the **principal filter** generated by A .

2) Let (X, τ) be a space and $x \in X$.

Then $\mathcal{U}(x)$ is a filter, the so called **neighbourhood filter** in $x \in X$.

Every neighbourhood base in $x \in X$ is a filter base generating $\mathcal{U}(x)$.

3) Let X be an infinite set. Then

$\mathcal{F} = \{F \subseteq X \;:\; X \setminus F \text{ is finite}\}$ is called the **Frechet filter**.

4) Let X be a set and (x_n) be a sequence in X .

Then $S_k = \{x_n \ : \ n \geq k\}$ is called the **kth tail** of (x_n) .

Obviously $\mathcal{B} = \{S_k \ : \ k \in \mathbb{N}\}$ is a filter base generating the so called **elementary filter** of (x_n)

$\mathcal{F} = \{F \subseteq X \ : \ \exists \, k \in \mathbb{N} \ \text{such that} \ S_k \subseteq F\}$

Thus we can assign to each sequence the corresponding elementary filter.

5) Let X, Y be sets, $f : X \to Y$ a function and $\mathcal{F}$ a filter on X .

Then $\{f(F) \ : \ F \in \mathcal{F}\}$ is a filter base on Y (in general not a filter).

The filter on Y generated by this filter base

$f(\mathcal{F}) = \{B \subseteq Y \ : \ \exists \, F \in \mathcal{F} \ \text{such that} \ f(F) \subseteq B\}$

is called the **image filter** with respect to $\mathcal{F}$ and $f : X \to Y$.

We now define the convergence of filters.

Definition. Let (X, τ) be a space, $\mathcal{F}$ a filter on X and $x \in X$.

1) $\mathcal{F}$ **converges to** $x \in$, $\mathcal{F} \to x$, if $\mathcal{U}(x) \subseteq \mathcal{F}$.

(We say that $\mathcal{F}$ is **finer** than $\mathcal{U}(x)$, resp. $\mathcal{U}(x)$ is **coarser** than $\mathcal{F}$.)

2) x is an **accumulation point of** $\mathcal{F}$, $x \in \mathrm{HP}(\mathcal{F})$, if every neighbourhood of x has nonempty intersection with each $F \in \mathcal{F}$,

i.e. if $x \in \overline{F} \quad \forall \, F \in \mathcal{F}$

Remarks.

1) $\mathrm{HP}(\mathcal{F}) = \bigcap\limits_{F \in \mathcal{F}} \overline{F}$

2) $\mathcal{U}(x) \to x$ for each $x \in X$

3) $\mathcal{F} \to x \quad \Rightarrow \quad x \in \mathrm{HP}(\mathcal{F})$

4) Let $x \in \mathrm{HP}(\mathcal{F})$. Then, as is easily seen, $\{U \cap F \ : \ U \in \mathcal{U}(x) \, , \ F \in \mathcal{F}\}$ is a filter base for a filter $\mathcal{G}$. Obviously, $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G} \to x$.

5) Let $\mathcal{F}$ be the elementary filter of the sequence (x_n) . Then

(i) $x \in \mathrm{HP}(\mathcal{F}) \quad \Leftrightarrow \quad x$ is accumulation point of (x_n) .

(ii) $\mathcal{F} \to x \quad \Leftrightarrow \quad (x_n) \to x$

The following results show that the notion of a filter is the appropriate generalization of the notion of a sequence in the context of topological spaces.

Proposition. Let (X, τ) be a space, $\emptyset \neq A \subseteq X$ and $x \in X$.

Then $x \in \overline{A}$ if and only if there is a filter $\mathcal{F}$ such that $\mathcal{F} \to x$ and $A \in \mathcal{F}$.

Proof. Let $x \in \overline{A}$. Then $U \cap A \neq \emptyset$ for each $U \in \mathcal{U}(x)$.

Hence $\{U \cap A \ : \ U \in \mathcal{U}(x)\}$ is a filterbase generating a filter $\mathcal{F}$.

Since $U \cap A \subseteq A$ and $U \cap A \subseteq U$ we have $A \in \mathcal{F}$ and $\mathcal{U}(x) \subseteq \mathcal{F}$, i.e. $\mathcal{F} \to x$.

Conversely, suppose that $x \notin \overline{A}$. Then there exists $U \in \mathcal{U}(x)$ such that $U \cap A = \emptyset$. Since $U \in \mathcal{F}$ and $A \in \mathcal{F}$ we have $U \cap A = \emptyset \in \mathcal{F}$, a contradiction. $\square$

Proposition. $f : (X, \tau) \to (Y, \sigma)$ is continuous at $x_0 \in X$ if and only if for each filter $\mathcal{F}$ on X with $\mathcal{F} \to x_0$ we have $f(\mathcal{F}) \to f(x_0)$.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous at $x_0 \in X$ and let $\mathcal{F} \to x_0$.

If $V \in \mathcal{U}(f(x_0))$ there exists $U \in \mathcal{U}(x_0)$ with $f(U) \subseteq V$.

Since $\mathcal{F} \to x_0$, $U \in \mathcal{F}$ and $f(U) \in f(\mathcal{F})$ and thus $V \in f(\mathcal{F})$. Hence $f(\mathcal{F}) \to f(x_0)$.

Conversely, we always have $\mathcal{U}(x_0) \to x_0$ and so, by assumption,

$$f(\mathcal{U}(x_0)) \to f(x_0) \text{ , i.e. } \mathcal{U}(f(x_0)) \subseteq f(\mathcal{U}(x_0)) \text{ .}$$

Hence, if $V \in \mathcal{U}(f(x_0))$ $\exists\, U \in \mathcal{U}(x_0)$ such that $f(U) \subseteq V$. $\square$

Proposition. A space (X,τ) is T_2 if and only if every filter on X converges to at most one point.

Proof. Let (X,τ) be T_2 and suppose for a filter $\mathcal{F}$ we have $\mathcal{F} \to x$ and $\mathcal{F} \to y$ where $x \neq y$.

By assumption, $\exists\, U \in \mathcal{U}(x)$, $V \in \mathcal{U}(y)$ with $U \cap V = \emptyset$.

Since $U, V \in \mathcal{F}$ we have $\emptyset = U \cap V \in \mathcal{F}$, a contradiction.

Now suppose that (X,τ) is **not** T_2 . Then there exist $x \neq y$ such that $U \cap V \neq \emptyset \ \ \forall\, U \in \mathcal{U}(x) \ \ V \in \mathcal{U}(y)$.

But then $\{U \cap V \ : \ U \in \mathcal{U}(x)\ ,\ V \in \mathcal{U}(y)\}$ is a filter base generating a filter $\mathcal{F}$ with $\mathcal{F} \to x$ and $\mathcal{F} \to y$. $\square$

Remark. We will see later that the notion of compactness can also be characterized via filters.

A space (X,τ) is compact if and only if every filter on X has an accumulation point if and only if every ultrafilter on X converges.

Proposition. Let $\mathcal{F}$ be a filter on $X = \prod_{i \in I} X_i$ and $x \in X$. Then

$$\mathcal{F} \to x \quad \Leftrightarrow \quad \forall\, i \in I \ : \ p_i(\mathcal{F}) \to x_i$$

Proof. "$\Rightarrow$" holds since each p_i is continuous.

"$\Leftarrow$": It is sufficient to show that every member of the canonical base containing x is a member of $\mathcal{F}$.

So let $x \in p_{i_1}^{-1}(O_{i_1}) \cap \ldots \cap p_{i_k}^{-1}(O_{i_k})$ where each O_{i_j} is open in X_{i_j} .

Then $x_{i_1} \in O_{i_1}$ and so, by assumption, $O_{i_1} \in p_{i_1}(\mathcal{F})$.

Hence there is $F_1 \in \mathcal{F}$ such that $p_{i_1}(F_1) \subseteq O_{i_1}$, i.e. $F_1 \subseteq p_{i_1}^{-1}(O_{i_1})$.

In the same manner we obtain $F_2, \ldots F_k \in \mathcal{F}$ with

$$F_2 \subseteq p_{i_2}^{-1}(O_{i_2}) \ , \ldots , \ F_k \subseteq p_{i_k}^{-1}(O_{i_k})$$

Thus $F_1 \cap \ldots F_k \subseteq p_{i_1}^{-1}(O_{i_1}) \cap \ldots \cap p_{i_k}^{-1}(O_{i_k})$.

Since $F_1 \cap \ldots F_k \in \mathcal{F}$ we have $p_{i_1}^{-1}(O_{i_1}) \cap \ldots \cap p_{i_k}^{-1}(O_{i_k}) \in \mathcal{F}$. $\square$

The set of all filters on X is partially ordered by the relation $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

Definition. A filter $\mathcal{U}$ on X is called **ultrafilter** if there is no filter $\mathcal{F}$ on X with $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{U} \neq \mathcal{F}$.

It follows easily from the lemma of Zorn that for each filter $\mathcal{F}$ on X there is at least one ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subseteq \mathcal{U}$.

By a previous remark we have

Proposition. Let $\mathcal{U}$ be an ultrafilter on X . Then

$$x \in \mathrm{HP}(\mathcal{U}) \quad \Rightarrow \quad \mathcal{U} \to x$$

Proposition. Let $\mathcal{U}$ be a filter on X . Then

$$\mathcal{U} \text{ is an ultrafilter} \quad \Leftrightarrow \quad \forall\, A \subseteq X \;:\; A \in \mathcal{U} \text{ or } X \setminus A \in \mathcal{U}$$

Proof.

"$\Rightarrow$": Suppose that $A \subseteq X$ and $A \notin \mathcal{U}$.

Then we have $(X \setminus A) \cap F \neq \emptyset$ for each $F \in \mathcal{U}$ (otherwise $F \subseteq A$ and $A \in \mathcal{U}$) .

Hence $\{(X \setminus A) \cap F \;:\; F \in \mathcal{U}\}$ is a filter base for a filter $\mathcal{G}$ with $X \setminus A \in \mathcal{G}$ and $\mathcal{U} \subseteq \mathcal{G}$.

Since $\mathcal{U}$ is an ultrafilter, $\mathcal{U} = \mathcal{G}$ and so $X \setminus A \in \mathcal{U}$.

"$\Leftarrow$": Suppose there is a filter $\mathcal{G}$ with $\mathcal{U} \subseteq \mathcal{G}$ and $\mathcal{U} \neq \mathcal{G}$.

Then there exists $A \subseteq X$ with $A \in \mathcal{G}$ but $A \notin \mathcal{U}$.

By assumption, $X \setminus A \in \mathcal{U} \subseteq \mathcal{G}$ and therefore $\emptyset = A \cap (X \setminus A) \in \mathcal{G}$, a contradiction. $\square$

Proposition.

Let $\mathcal{U}$ be an ultrafilter on X and $f : X \to Y$ a function.

Then $f(\mathcal{U})$ is an ultrafilter.

Proof. Let $B \subseteq Y$. By a previous result,

$$f^{-1}(B) \in \mathcal{U} \quad \text{or} \quad X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \in \mathcal{U} \ .$$

If $f^{-1}(B) \in \mathcal{U}$ then $B \in f(\mathcal{U})$ since $f(f^{-1}(B)) \subseteq B$.

If $f^{-1}(Y \setminus B) \in \mathcal{U}$ then $Y \setminus B \in f(\mathcal{U})$ since $f(f^{-1}(Y \setminus B)) \subseteq Y \setminus B$.

Hence $f(\mathcal{U})$ is an ultrafilter. $\square$

Remark. Let X be a set and $x \in X$. Then

$$\mathcal{F} = \{F \subseteq X \ : \ x \in F\} \text{ is an ultrafilter.}$$

Observe that $\bigcap_{F \in \mathcal{F}} F = \{x\}$.

Filters $\mathcal{F}$ with the property that $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ are called **fixed**, otherwise **free** . The Frechet filter on an infinite set is an example of a free filter.