10. Covering properties

Definition. A space (X, τ) is called

1) **compact** if every open cover of X contains a finite subcover, i.e. if $X = \bigcup_{i \in I} O_i$, where each O_i is open, then there exist $i_1, i_2, \ldots, i_k \in I$ such that $X = O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_k}$.

2) **Lindelöf** if every open cover of X contains a countable subcover, i.e. if

$X = \bigcup_{i \in I} O_i$, where each O_i is open, then $X = \bigcup_{n \in \mathbb{N}} O_{i_n}$.

3) **countably compact** if every countable open cover of X contains a finite subcover.

Definition. Let (X, τ) be a space. $A \subseteq X$ is called a **compact** (resp. Lindelöf, countably compact) **subset** if the subspace $(A, \tau|_A)$ is compact (resp. Lindelöf, countably compact).

This is, in the compact case, equivalent to:

if $A \subseteq \bigcup_{i \in I} O_i$, where each O_i is open in X , then there exist $i_1, i_2, \ldots, i_k \in I$ such that $A \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_k}$.

(Similar for Lindelöf and countably compact.)

Remark. Obviously,

compact $\Rightarrow$ Lindelöf , and compact $\Rightarrow$ countably compact.

It follows also from the definition that if (X, τ) is compact (resp. Lindelöf, countably compact) and $\sigma \subseteq \tau$, then (X, σ) is compact (resp. Lindelöf, countably compact).

Remark. Consider $X = \mathbb{R}^n$ (or $X = \mathbb{C}^n$) with the usual topology.

It is shown in calculus (**Theorem of Heine-Borel**) that $C \subseteq X$ is compact if and only if C is closed and bounded (with respect to the norm).

Hence $\mathbb{R}^n$ is **not** compact but each closed ball $B(x,r) = \{y \in \mathbb{R}^n : \|x-y\| \leq r\}$ is compact.

We observe also that the Sorgenfrey line cannot be compact (since the usual topology on $\mathbb{R}$ is coarser and not compact).

However, the Sorgenfrey line is hereditarily Lindelöf, i.e. every subspace is Lindelöf (Exercise).

Proposition. (X,τ) is countably compact if and only if every sequence has an accumulation point.

Proof.

Suppose that (X,τ) is countably compact and let (x_n) be a sequence.

The set $\mathrm{HP}(x_n)$ of accumulation points of (x_n) is $\mathrm{HP}(x_n) = \bigcap_{k\in\mathbb{N}} \overline{S_k}$, where $S_k = \{x_n : n \geq k\}$. Observe that $S_m \subseteq S_k$ whenever $m \geq k$.

If $\mathrm{HP}(x_n) = \emptyset$ then the open cover $X = \bigcup_{k\in\mathbb{N}} (X \setminus \overline{S_k})$ has a finite subcover $X = (X \setminus \overline{S_{k_1}}) \cup \ldots \cup (X \setminus \overline{S_{k_r}})$ and so $\overline{S_{k_1}} \cap \ldots \cap \overline{S_{k_r}} = \emptyset$ which is not possible.

Therefore (x_n) must have an accumulation point.

Now suppose that every sequence has an accumulation point but (X,τ) is not countably compact. Then there is a countable open cover (O_n) of X without a finite subcover.

For each $n \in \mathbb{N}$ let $V_n = O_1 \cup \ldots \cup O_n$. Then $V_n \neq X$ for each n, $V_n \subseteq V_m$ whenever $n \leq m$ and $X = \bigcup_{n\in\mathbb{N}} V_n$.

For each $n \in \mathbb{N}$ pick $x_n \in X \setminus V_n$.

Observe that for the sequence (x_n) we have $S_k \subseteq X \setminus V_k$, since $X \setminus V_n \subseteq X \setminus V_k$ for each $n \geq k$.

Since $X \setminus V_k$ is closed, we have $\overline{S_k} \subseteq X \setminus V_k$ for each k .

By assumption, there exists an accumulation point x of (x_n) . It follows that $x \notin V_k$ for each k which is not possible. $\square$

Example. Previously we showed that there is an uncountable well ordered set $(X, <)$ where each element has at most countably many predecessors. Let X have the order topology τ .

Then (X, τ) is **not** compact since the open cover $X = \bigcup_{y \in X} [0, y)$ has no finite subcover.

However, one can show that (X, τ) is countably compact.

(Idea of proof: First observe that every sequence has a supremum. For a sequence (a_n) the set $\bigcup_{n \in \mathbb{N}} [0, a_n)$ is countable therefore $\exists\ x \in X$ such that $x \notin \bigcup_{n \in \mathbb{N}} [0, a_n)$, i.e. $a_n < x \ \ \forall\ n \in \mathbb{N}$. So there exists an upper bound for (a_n) . Since X is well ordered, we have a smallest upper bound.

Suppose we have a countable open cover $\{O_k \ : \ k \in \mathbb{N}\}$ of X having no finite subcover. We may assume that $O_k \subseteq O_{k+1} \ \ \forall\ n$.

Inductively one can construct a sequence (a_n) with the following properties.

$[0, a_n]$ is covered by finitely many O_k . a_{n+1} is the smallest element not contained in the union of those O_k and O_n . Considering the supremum of (a_n) leads to a contradiction.)

Remark. For metric spaces (X, d) with topology τ_d we mention without proof:

compact $\Leftrightarrow$ countably compact $\Leftrightarrow$ Every sequence contains a convergent subsequence

Proposition. Let (X, τ) be a space, $C \subseteq X$ compact (resp. Lindelöf, countably compact), $A \subseteq X$ closed with $A \subseteq C$.

Then A is compact (resp. Lindelöf, countably compact).

Proof. Let $A \subseteq \bigcup_{i \in I} O_i$, where each O_i is open in X .

Then $C \subseteq \bigcup_{i \in I} O_i \cup (X \setminus A)$ and, since $X \setminus A$ is open, there exist $i_1, i_2, \ldots, i_k \in I$ such that $C \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_k} \cup (X \setminus A)$.

Hence $A \subseteq O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_k}$. $\square$

Remarks.

(i) **Closed** subspaces of compact (resp. Lindelöf, countably compact) spaces are compact (Lindelöf, countably compact).

(ii) Consequently, a closed discrete subspace of a compact (resp. Lindelöf) space must be finite (resp. at most countable).

Therefore, the Niemitzky plane is not Lindelöf.

(iii) From a theorem in the chapter about bases it follows that every second countable space is Lindelöf.

Since every subspace of a second countable space is second countable, a second countable space is hereditarily Lindelöf.

Observe that the Sorgenfrey line is hereditarily Lindelöf but not second countable.

For metric spaces, however, we have the following important result.

Theorem. Let (X, d) be a metric space with topology τ_d . Then the following are equivalent:

1) (X, τ_d) is second countable,

2) (X, τ_d) is Lindelöf,

3) (X, τ_d) is **separable**, i.e. there is a countable dense subset $D \subseteq X$ (with $\overline{D} = X$).

Proof.

1) $\Rightarrow$ 2) always holds.

2) $\Rightarrow$ 3): For each $n \in \mathbb{N}$, $X = \bigcup_{x\in X} K(x, \frac{1}{n})$ is an open cover.

By assumption there exists a countable subset $D_n \subseteq X$ such that $X = \bigcup_{x\in D_n} K(x, \frac{1}{n})$.

If $D = \bigcup_{n\in\mathbb{N}} D_n$, then D is countable. We claim that $\overline{D} = X$.

Suppose $\exists\, y \in X$ with $y \notin \overline{D}$ $\Rightarrow$ $\exists\, m \in \mathbb{N}$ with $K(y, \frac{1}{m}) \cap D = \emptyset$.
$\exists\, x \in D_m \subseteq D$ such that $y \in K(x, \frac{1}{m})$.
Hence $x \in K(y, \frac{1}{m})$, a contradiction. Therefore, $\overline{D} = X$.

3) $\Rightarrow$ 1): Let D be a **countable** subset with $\overline{D} = X$.
Let $\mathcal{B} = \{K(x, \frac{1}{2^n}) \ : \ x \in D \ , \ n \in \mathbb{N}\}$.
Then $\mathcal{B}$ is a countable (!) family of open sets. We claim that $\mathcal{B}$ is a base.

Let $O \subseteq X$ be open and $y \in O$. Then $\exists\, n \in \mathbb{N}$ with $K(y, \frac{1}{2^n}) \subseteq O$.
Choose $x \in D \cap K(y, \frac{1}{2^{n+1}})$.
Because of the triangle inequaltiy we have

$$y \in K(x, \tfrac{1}{2^{n+1}}) \subseteq K(y, \tfrac{1}{2^n}) \subseteq O . \quad \square$$

Compactness can also be characterized via filters.

Theorem. For a space (X, τ) the following are equivalent:

1) (X, τ) is compact
2) Every filter on X has an accumulation point
3) Every ultrafilter on X converges.

Proof.

1) $\Rightarrow$ 2): Let $\mathcal{F}$ be a filter on X and suppose that $\mathcal{F}$ has no accumulation point, i.e. $\bigcap_{F\in\mathcal{F}} \overline{F} = \emptyset$.

Then $X = \bigcup_{F\in\mathcal{F}} (X \setminus \overline{F})$ is an open cover of X .

By assumption, there exist $F_1, \ldots, F_k \in \mathcal{F}$ such that

$$X = (X \setminus \overline{F_1}) \cup \ldots \cup (X \setminus \overline{F_k}) .$$

Hence $\overline{F_1} \cap \ldots \cap \overline{F_k} = \emptyset$ and so $F_1 \cap \ldots \cap F_k = \emptyset$, a contradiction.

2) $\Rightarrow$ 3): Let $\mathcal{U}$ be an ultrafilter on X . By hypothesis, there exists an accumulation point $x \in X$ of $\mathcal{U}$.

Previously we showed that in such a case $\mathcal{U} \to x$.

3) $\Rightarrow$ 2): Let $\mathcal{F}$ be a filter on X . Then there is an ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subseteq \mathcal{U}$.

By hypothesis, $\exists\, x \in X$ such that $\mathcal{U} \to x$.

Since $x \in \bigcap_{U\in\mathcal{U}} \overline{U} \subseteq \bigcap_{F\in\mathcal{F}} \overline{F}$, x is an accumulation point of $\mathcal{F}$.

2) $\Rightarrow$ 1): We assume that (X, τ) is not compact. Then there is an open cover $\{O_i \ : \ i \in I\}$ of X having no finite subcover.

Hence $\forall\, I' \subseteq I$, I' finite , we have $X \neq \bigcup_{i\in I'} O_i$ resp.

$F_{I'} = \bigcap_{i\in I'} (X \setminus O_i) \neq \emptyset$. Observe that $F_{I'}$ is closed.

The family $\mathcal{B} = \{F_{I'} \ : \ I' \subseteq I \ , \ I' \text{ finite}\}$ is obviously a filter base,

since $F_{I'} \cap F_{I''} = F_{I'\cup I''}$.

Let $\mathcal{F}$ be the filter generated by $\mathcal{B}$. By hypothesis, $\mathcal{F}$ has an accumulation point $x \in X$.

There exists $i_0 \in I$ such that $x \in O_{i_0}$. Let $I^* = \{i_0\}$.

Then $F_{I^*} = X \setminus O_{i_0} = \overline{F_{I^*}} \in \mathcal{F}$, hence $x \in X \setminus O_{i_0}$, a contradiction. □

Without the (not so easy) proof we mention another important characterization of compactness, known as the **Alexander subbase theorem**.

Theorem. A space (X,τ) is compact if and only if every cover of members by a subbase $\mathcal{S}$ of (X,τ) has a finite subcover.

Next we consider the relationships between covering properties and separation axioms.

Theorem. Let (X,τ) be T_2 , $C \subseteq X$ be compact and $x \notin C$.

Then there exist open sets O_1 , O_2 such that $x \in O_1$, $C \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$.

Proof. For each $y \in C$ there are open sets V_y, W_y such that

$$y \in V_y \ , \ x \in W_y \ \text{ and } \ V_y \cap W_y = \emptyset \ .$$

Since $C \subseteq \bigcup_{y \in C} V_y$ there exist $y_1, \ldots, y_k \in C$ such that

$$C \subseteq V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_k} \ .$$

If $O_2 = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_k}$ and $O_1 = W_{y_1} \cap W_{y_2} \cap \ldots \cap W_{y_k}$ then O_1 and O_2 are open, $x \in O_1$, $C \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$. □

Corollary. In a T_2-space, compact subsets are closed.

(Note that if τ is the cofinite topology on a infinite set X then every subspace of (X,τ) is compact but not closed.)

Corollary.

Let (X,τ) be T_2 , C_1 , $C_2 \subseteq X$ be compact and $C_1 \cap C_2 = \emptyset$.

Then $\exists\, O_1, O_2 \in \tau$ such that $C_1 \in O_1$, $C_2 \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$.

Proof. With the previous theorem, for each $y \in C_2$ there exist open sets V_y, W_y such that $y \in V_y$, $C_1 \subseteq W_y$ and $V_y \cap W_y = \emptyset$.

Since $C_2 \subseteq \bigcup_{y \in C} V_y$ there exist $y_1, \ldots, y_k \in C$ such that

$C_2 \subseteq V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_k}$.

If $O_2 = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_k}$ and $O_1 = W_{y_1} \cap W_{y_2} \cap \ldots \cap W_{y_k}$ then O_1 and O_2 are open, $C_1 \subseteq O_1$, $C_2 \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$. $\square$

Theorem. Every compact T_2−space (X, τ) is normal.

Proof. Let $A, B \subseteq X$ be closed and disjoint. Then A, B are also compact.

By the previous Corollary, there exist open sets O_1 , O_2 with $A \subseteq O_1$, $V \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$. $\square$

Corollary. Every subspace of a compact T_2−space (X, τ) is completely regular.

Remark. One can show that every regular Lindelöf space is also normal. Therefore the Sorgenfrey line is normal (since it is easy to see that the Sorgenfrey line is regular).

Certain covering properties "behave well" with respect to continuity.

Theorem. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and let $A \subseteq X$ be compact (resp. Lindelöf, countably compact).

Then $f(A) \subseteq Y$ is compact (resp. Lindelöf, countably compact).

Proof. (for the compact case)

Let $f(A) \subseteq \bigcup_{i \in I} V_i$ where each $V_i \in \sigma$. Then $A \subseteq \bigcup_{i \in I} f^{-1}(V_i)$, so there exist $i_1, i_2, \ldots, i_k \in I$ such that $A \subseteq f^{-1}(V_{i_1}) \cup f^{-1}(V_{i_2}) \cup \ldots \cup f^{-1}(V_{i_k})$.

Hence $f(A) \subseteq V_{i_1} \cup V_{i_2} \cup \ldots \cup V_{i_k}$. $\square$

Corollary. In particular, if $f : (X, \tau) \to \mathbb{R}$ is continuous and $A \subseteq X$ is compact, then $f(A) \subseteq \mathbb{R}$ is closed and bounded.

Therefore there exist $a_0, a_1 \in A$ such that

$f(a_0) = \min\{f(a) \ : \ a \in A\}$ and

$f(a_1) = \max\{f(a) \ : \ a \in A\}$

Proposition. Let (X, τ) be compact, (Y, σ) be T_2 and $f : (X, \tau) \to (Y, \sigma)$ be continuous.

Then f is a closed function.

Proof. $A \subseteq X$ is closed $\Rightarrow$ $A \subseteq X$ is compact $\Rightarrow$ $f(A) \subseteq Y$ is compact $\Rightarrow$ $f(A) \subseteq Y$ is closed. $\square$

Remark. If f is, in addition, bijective then f is a homeomorphism!

We now consider a family (X_i, τ_i) , $i \in I$ of spaces and the product space $X = \prod_{i \in I} X_i$.

Since the projections are continuous and surjective we conclude that if $X = \prod_{i \in I} X_i$ is compact then all the spaces (X_i, τ_i) are also compact.

The converse of this observation is a fundamental theorem in topology with widespread applications, known as **Tychonoff's theorem** .

Using several previous results we are able to give a short proof.

Theorem. (**Tychonoff**)

For each $i \in I$ let (X_i, τ_i) be compact. Then $X = \prod_{i \in I} X_i$ is compact.

Proof. Let $\mathcal{U}$ be an ultrafilter on X . Then $p_i(\mathcal{U})$ is an ultrafilter on X_i for each $i \in I$.

Since each X_i is compact, for each $i \in I$ there exists $x_i \in X_i$ such that $p_i(\mathcal{U}) \to x_i$.

By a previous result, $\mathcal{U} \to x$. Therefore $X = \prod_{i \in I} X_i$ is compact. $\square$

Corollary. The set of all functions $f : \mathbb{R} \to [0, 1]$ with the topology of pointwise convergence is a compact Hausdorff space, since it is $[0, 1]^{\mathbb{R}}$.

It is clear that the class of compact spaces plays a fundamental role in topology and analysis. However, many interesting spaces are not compact globally but in a local manner.

Definition. A space (X, τ) is called **locally compact** if each $x \in X$ has a compact neighbourhood.

Remarks.

(i) Every compact space is locally compact.

(ii) $\mathbb{R}^n$ is locally compact but not compact. The discrete topology on an infinite set is locally compact but not compact.

In the following we will show that each locally compact T_2 space can be embedded into a compact T_2 space in a special way.

Let (X, τ) be locally compact, T_2 and not compact.

Let $a \notin X$ and let $X^* = X \cup \{a\}$. We add one point to X , obtain in this way the set X^* and construct a topology on X^* .

Let τ^* be the family of all subsets $V \subseteq X^*$ such that

either $V \subseteq X$ and $V \in \tau$ or

$a \in V$ and $X \setminus V$ is compact in (X, τ) .

Observe that $\tau \subseteq \tau^*$ and that $V \in \tau^* \Rightarrow V \cap X \in \tau$.

We now show that τ^* is a topology on X^* .

$\emptyset \in \tau^*$ since $\emptyset \in \tau \subseteq \tau^*$.

$X^* \in \tau^*$ since $X \setminus X^* = \emptyset$ is compact in (X, τ) .

Let $V_1, V_2 \in \tau^*$.

If $V_1, V_2 \subseteq X$ then $V_1, V_2 \in \tau$ and $V_1 \cap V_2 \in \tau \subseteq \tau^*$.

If $a \in V_1$ and $V_2 \subseteq X$ then $V_1 \cap V_2 = (V_1 \cap X) \cap V_2 \in \tau \subseteq \tau^*$.

Similar if $V_1 \subseteq X$ and $a \in V_2$.

If $a \in V_1 \cap V_2$ then $X \setminus (V_1 \cap V_2) = (X \setminus V_1) \cup (X \setminus V_2)$ is the union of two compact subsets of (X, τ) and therefore compact. Hence $V_1 \cap V_2 \in \tau^*$.

Now let $V_i \in \tau^*$ for each $i \in I$.

If $V_i \subseteq X$ for each $i \in I$ then $\bigcup\limits_{i \in I} V_i \in \tau \subseteq \tau^*$.

Otherwise there exists $i_0 \in I$ such that $a \in V_{i_0}$. Then $X \setminus V_{i_0}$ is compact and closed in (X, τ) .

Then $X \setminus (\bigcup\limits_{i \in I} V_i)$ is a closed (in (X, τ)) subset of the compact set $X \setminus V_{i_0}$ and therefore itself compact. Consequently $\bigcup\limits_{i \in I} V_i \in \tau^*$.

Hence τ^* is a topology on X^* .

Definition. The space (X^*, τ^*) is called the **1-point-compactification** of (X, τ) .

Remark. It is obvious that $\tau^*|_X = \tau$ therefore (X, τ) is a subspace of (X^*, τ^*) and the inclusion function $j : X \to X^*$ is an embedding.

We now show that (X^*, τ^*) is compact and T_2 .

Let $X^* = \bigcup\limits_{i \in I} V_i$ where $V_i \in \tau^*$ for each $i \in I$.

Choose $i_0 \in I$ with $a \in V_{i_0}$. Then $X \setminus V_{i_0} \subseteq \bigcup\limits_{i \in I} (X \cap V_i)$.

$\exists\ i_1, \ldots, i_k \in I$ such that $X \setminus V_{i_0} \subseteq (X \cap V_{i_1}) \cup \ldots \cup (X \cap V_{i_k})$.

Hence $X^* = V_{i_0} \cup V_{i_1} \cup \ldots \cup V_{i_k}$ showing that (X^*, τ^*) is compact .

Now let $x, y \in X^*$ with $x \neq y$.

If $x \neq a$ and $y \neq a$ then there exist disjoint open neighbourhoods as (X, τ) is T_2 .

So let $x \in X$ and $y = a$.

Let $W \subseteq X$ be a compact neighbourhood of x (with respect to (X, τ)) and let $V = X^* \setminus W$.

Then $a \in V$ and $V \in \tau^*$. Obviously, W and V are the required disjoint neighbourhoods and therefore (X^*, τ^*) is T_2 .

Remark. According to a previous result (X^*, τ^*) is also normal, and therefore every locally compact T_2 space is completely regular.

Example. It is easily checked that the 1-point-compactification of $\mathbb{R}$ is homeomorphic to S^1 and the 1-point-compactification of $\mathbb{R}^2$ is homeomorphic to S^2 .