11. Connected Spaces

Definition. Let (X, τ) be a topological space.

1) (X, τ) is called **connected** if X can **not** be represented as the union of two disjoint nonempty open sets.

2) $A \subseteq X$ is called connected (or a connected subspace) if $(A, \tau|_A)$ is connected.

Remarks.

(i) (X, τ) is connected if and only if $\emptyset$ and X are the only subsets that are both open and closed.

(ii) $C = \{x\}$ is connected for each $x \in X$.

(iii) $A \subseteq X$ is connected $\Leftrightarrow$ whenever $A \subseteq O_1 \cup O_2$ where O_1, O_2 are nonempty open sets and $A \cap O_1 \cap O_2 = \emptyset$ it follows that $A \cap O_1 = \emptyset$ or $A \cap O_2 = \emptyset$.

(iv) $A = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$ is not connected.

($A \subseteq O_1 \cup O_2$ with $O_1 = (-\frac{1}{2}, \frac{3}{2})$ and $O_2 = (\frac{3}{2}, \frac{7}{2})$ and (iii))

Proposition. A subset $C \subseteq \mathbb{R}$ with $|C| > 1$ is connected if and only if C is a convex subset (i.e. an interval).

Proof.

"$\Rightarrow$": Suppose that C is not convex. Then $\exists\, x, y \in C$ $\exists\, z$ such that $x < z < y$ and $z \notin C$.

Let $O_1 = (-\infty, z)$ and $O_2 = (z, \infty)$. Then O_1, O_2 are nonempty and open. Moreover, $C \subseteq O_1 \cup O_2$ and $C \cap O_1 \cap O_2 = \emptyset$.

However, $x \in C \cap O_1 \neq \emptyset$ and $y \in C \cap O_2 \neq \emptyset$, a contradiction. So C is convex.

"$\Leftarrow$": Now let C be convex and suppose that C is not connected.

Then there are nonempty open sets $O_1, O_2 \subseteq \mathbb{R}$ with $C \subseteq O_1 \cup O_2$, $C \cap O_1 \cap O_2 = \emptyset$ and $C \cap O_1 \neq \emptyset$ $C \cap O_2 \neq \emptyset$.

Pick $x \in C \cap O_1$, $y \in C \cap O_2$, wlog $x < y$.

By hypothesis, $[x, y] \subseteq C \subseteq O_1 \cup O_2$.

Let $z = \sup\{O_1 \cap [x, y]\}$. Then $z \in [x, y]$.

If $z \in O_1$ then $z \neq y$. So $\exists\ \varepsilon > 0$ with $[z, z + \varepsilon) \subseteq O_1 \cap [x, y]$ and therefore $\exists\ z^* \in O_1 \cap [x, y]$ with $z < z^*$, a contradiction to $z = \sup\{O_1 \cap [x, y]\}$.

If $z \in O_2$ then $z \neq x$. So $\exists\ \varepsilon > 0$ with $(z - \varepsilon, z] \subseteq O_2 \cap [x, y]$. This is again a contradiction to $z = \sup\{O_1 \cap [x, y]\}$. So C is connected. $\square$

Theorem. Let (X, τ) be a space, $C_0 \subseteq X$ be connected and for each $i \in I$ let $C_i \subseteq X$ be connected such that $C_0 \cap C_i \neq \emptyset$ for each $i \in I$.

Then $C = C_0 \cup \bigcup_{i \in I} C_i$ is connected.

Proof. Suppose that C is not connected. Then there exist open sets $O_1, O_2 \subseteq X$ with $C \subseteq O_1 \cup O_2$, $C \cap O_1 \cap O_2 = \emptyset$ and $C \cap O_1 \neq \emptyset$ and $C \cap O_2 \neq \emptyset$.

For each $i \in I$ we have $C_i \subseteq O_1 \cup O_2$ and $C_i \cap O_1 \cap O_2 = \emptyset$.

Therefore $C_i \cap O_1 = \emptyset$ or $C_i \cap O_2 = \emptyset$.

We also have $C_0 \cap O_1 = \emptyset$ or $C_0 \cap O_2 = \emptyset$, wlog we assume that $C_0 \cap O_2 = \emptyset$.

Then $C_0 \subseteq O_1$ and $\emptyset \neq C_0 \cap C_i \subseteq O_1 \cap C_i$. And so $C_i \cap O_2 = \emptyset$ for each $i \in I$. Therefore $C \cap O_2 = \emptyset$, a contradiction. $\square$

Proposition.

1) Let (X, τ) be a space and let $A \subseteq X$ be connected. Then $\overline{A}$ is connected.

2) Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and let $C \subseteq X$ be connected.

Then $f(C) \subseteq Y$ is connected.

Proof. Ad 1) Proof as exercise.

Ad 2) Let $f(C) \subseteq V_1 \cup V_2$ where V_1, V_2 are open in (Y, σ) and $f(C) \cap V_1 \cap V_2 = \emptyset$.

Then $C \subseteq f^{-1}(V_1) \cup f^{-1}(V_2)$ and $C \cap f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$.

Since C is connected, either $C \cap f^{-1}(V_1) = \emptyset$ or $C \cap f^{-1}(V_2) = \emptyset$.

Hence either $f(C) \cap V_1 = \emptyset$ or $f(C) \cap V_2 = \emptyset$, so $f(C)$ is connected. $\square$

Remarks.

1) Let $\alpha : [0,1] \to (X, \tau)$ be a continuous path. Then $\alpha([0,1])$ is a connected subset of (X, τ) .

In particular, if $\alpha : [0, 2\pi] \to \mathbb{R}$ with $\alpha(t) = \sin t$ then the function $f : [0, 2\pi] \to \mathbb{R}^2$ with $f(t) = (t, \alpha(t))$ is continuous. Hence the graph of the Sinus-function is a connected subset of $\mathbb{R}^2$.

2) (**Intermediate value theorem**)

Let $f : (X, \tau) \to \mathbb{R}$ be continuous and let $C \subseteq X$ be connected. Let $x, y \in C$ such that $f(x) < f(y)$.

Then for each $t \in \mathbb{R}$ with $f(x) < t < f(y)$ there exists $a_t \in C$ such that $f(a_t) = t$.

Proof. $f(C) \subseteq \mathbb{R}$ is connected, therefore $[f(x), f(y)] \subseteq f(C)$. $\square$

3) The Sorgenfrey line is not connected.

4) If (X, τ) is connected and $\sigma \subseteq \tau$ then (X, σ) is connected.

5) Let $C \subseteq \mathbb{R}^n$ be a star domain, i.e. there exists $a \in C$ such that with each $x \in C$ the connecting line from a to x is contained in C .

Then C is connected. (Proof as exercise)

6) $\mathbb{R}^n \setminus \{0\}$ is connected. (Proof as exercise)

Let (X, τ) be a space and $x \in X$. Let

$$C_x = \bigcup \{C \subseteq X \ : \ C \text{ is connected and } x \in C\}$$

Since $\{x\}$ is connected it follows that C_x is the largest connected subset that contains x.

C_x is called the **connected component** of x.

Now the following holds (proof as exercise):

(i) (X, τ) is connected $\Rightarrow$ $C_x = X \quad \forall\, x \in X$

(ii) $\forall\, x, y \in X \ : \ C_x \cap C_y = \emptyset$ or $C_x = C_y$

(iii) C_x is closed $\forall\, x \in X$

(iv) If (X, τ) is the Sorgenfrey line then $C_x = \{x\} \ \ \forall\, x \in X$

A stronger version of connectedness is provided by the notion of **path connectedness**.

Definition. Let (X, τ) be a space and $x, y \in X$.

A (continuous) **path** from x to y is a continuous function $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$.

Definition.

1) (X, τ) is called **path connected** if for all $x, y \in X$ there is a continuous path from x to y.

2) $C \subseteq X$ is called **path connected** if for all $x, y \in C$ there is a continuous path $\alpha : [0,1] \to C$ with $\alpha(0) = x$ and $\alpha(1) = y$.

Remark. Obviously, $\mathbb{R}^n$, every convex subset of $\mathbb{R}^n$ and $[0,1] \subseteq \mathbb{R}$ are path connected.

(For $x, y \in \mathbb{R}^n$ consider $\alpha(t) = (1-t)x + ty$)

Proposition. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and let $C \subseteq X$ be path connected. Then $f(C) \subseteq Y$ is path connected.

Proof. Let $y_0, y_1 \in f(C)$. Pick $x_0, x_1 \in C$ such that $y_0 = f(x_0)$ and $y_1 = f(x_1)$.

By assumption there is a continuous function $\alpha : [0,1] \to X$ with $\alpha([0,1]) \subseteq C$ and $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Then $\beta = f \circ \alpha : [0,1] \to Y$ is the required path in $f(C)$ from y_0 to y_1 . $\square$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $F : \mathbb{R} \to \mathbb{R}^2$ with $F(t) = (t, f(t))$ is also continuous. Therefore $F(\mathbb{R})$, i.e. the graph of f , is a path connected subset of $\mathbb{R}^2$.

Theorem. If (X, τ) is path connected then (X, τ) is connected.

Proof. Let $X = O_1 \cup O_2$ where O_1, O_2 are open and $O_1 \cap O_2 = \emptyset$. We assume that $O_1 \neq \emptyset$ and $O_2 \neq \emptyset$.

Pick $x \in O_1$ and $y \in O_2$. Then there exists a continuous function $\alpha : [0,1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$.

Then $[0,1] = \alpha^{-1}(O_1) \cup \alpha^{-1}(O_2)$ with $\alpha^{-1}(O_1) \cap \alpha^{-1}(O_2) \cap [0,1] = \emptyset$.

However, $0 \in \alpha^{-1}(O_1) \neq \emptyset$ and $1 \in \alpha^{-1}(O_2) \neq \emptyset$, which contradicts the fact that $[0,1]$ is connected.

Thus $O_1 = \emptyset$ or $O_2 = \emptyset$ showing that (X, τ) is connected. $\square$

Example.

Let $A = \{(x, \sin\frac{1}{x}) \ : \ 0 < x \leq 1\} \cup (\{0\} \times [-1,1]) \subseteq \mathbb{R}^2$.

If $A^* = \{(x, \sin\frac{1}{x}) \ : \ 0 < x \leq 1\}$ then A^* is path connected (by a previous consideration) and thus connected.

Since $A = \overline{A^*}$, A is connected.

However, one can show that A is not path connected (there is no continuous path from a point from A^* to a point of $\{0\} \times [-1,1]$).

In concluding we mention some facts whose proofs are left as an exercise.

Proposition.

1) Let (X, τ) be a space and let $x, y, z \in X$.

If there exists a continuous path from x to y and a continuous path from y to z then there exists a continuous path from x to z .

2) Let $C_0 \subseteq X$ be path connected and for each $i \in I$ let $C_i \subseteq X$ be path connected such that $C_0 \cap C_i \neq \emptyset$ for each $i \in I$.

Then $C = C_0 \cup \bigcup_{i \in I} C_i$ is path connected.

3) $\mathbb{R} \setminus \{0\}$ is obviously not path connected but $\mathbb{R}^n \setminus \{0\}$ for $n > 1$ is.

Remark. Two continuous paths $\alpha, \beta : [0,1] \to X$ from x to y are called **homotopic** if there is a continuous function $F : [0,1] \times [0,1] \to X$ such that $F(t,0) = \alpha(t)$ and $F(t,1) = \beta(t)$ for each $t \in [0,1]$, and $F(0,s) = x$ and $F(1,s) = y$ for each $s \in [0,1]$

Observe that for each $s \in [0,1]$ we get a continuous path $t \mapsto F(t,s)$.

For $s = 0$ we get α , for $s = 1$ we get β . We can say that α gets continuously "deformated" into β .

A **loop** (in $x \in X$) is a continuous path $\alpha : [0,1] \to X$ with $\alpha(0) = \alpha(1) = x$.

Observe that there is always the trivial (or constant) loop with the property $\alpha(t) = x \quad \forall\, t \in [0,1]$.

This leads to another stronger notion of connectedness. A space (X, τ) is called **simply connected** if it is path connected and for each $x \in X$, every loop in $x \in X$ is homotopic to the constant loop in $x \in X$.