12. Compactifications

A fundamental historical problem was: characterize the topological spaces (X, τ) that can be embedded in a compact T_2-space.
And related: characterize (in a useful way) the subspaces of compact T_2-spaces.

We first observe that any compact T_2-space is normal and thus completely regular. As a consequence every such space (X, τ) must necessarily be completely regular.

We also observe that for locally compact T_2-spaces we have constructed such an embedding and obtained the 1-point-compactification.

In the following we want to embed a space in a certain product space. A fundamental tool will be the **diagonal lemma**.

Let (X, τ) be a space, let (Y_i, σ_i) be a space for each $i \in I$ and let $\mathcal{A} = \{f_i : (X, \tau) \to (Y_i, \sigma_i) \ : \ i \in I\}$ be a family of continuous functions.

Then $\mathcal{A}$ induces a continuous function $e : X \to \prod_{i \in I} Y_i$ by

$(e(x))_i = f_i(x)$ i.e. $p_i \circ e = f_i$

(the f_i are the component functions of e)

Definition.

(i) $\mathcal{A}$ is called **point-separating** if for each $x, y \in X$ with $x \neq y$ there exists $i \in I$ with $f_i(x) \neq f_i(y)$.

(ii) $\mathcal{A}$ is said to **separate points from closed sets** if for each closed subset $A \subseteq X$ and each $x \notin A$ there exists $i \in I$ such that $f_i(x) \notin \overline{f_i(A)}$.

Theorem. (**Diagonal lemma**)

If $\mathcal{A} = \{f_i : (X, \tau) \to (Y_i, \sigma_i) \ : \ i \in I\}$ is a family of continuous functions that is point-separating and separating points from closed sets, then $e : X \to \prod_{i \in I} Y_i$ is an embedding.

Proof. We already observed that e is continuous.

Now we show that e is injective. Let $x, y \in X$ with $x \neq y$. Then there is $i \in I$ with $f_i(x) \neq f_i(y)$ and so $e(x) \neq e(y)$.

To complete the proof we need to show that $e : X \to e(X)$ is an open function.

Let $U \subseteq X$ be open and $z \in e(U)$. Then there is $x \in U$ with $z = e(x)$, and $x \notin X \setminus U$ and $X \setminus U$ is closed.

By assumption, there is $i \in I$ with

$$f_i(x) \notin \overline{f_i(X \setminus U)} \quad \text{resp.} \quad f_i(x) = Y_i \setminus \overline{f_i(X \setminus U)}$$

Let $W = p_i^{-1}(Y_i \setminus \overline{f_i(X \setminus U)})$. Then W is open in $\prod_{i \in I} Y_i$.

Since $z_i = f_i(x)$, $z \in W \cap e(X)$.

Observe that $W \cap e(X)$ is a neighbourhood of z with respect to the subspace $e(X)$.

We claim that $W \cap e(X) \subseteq e(U)$. Let $z' \in W \cap e(X)$. Then there is $x' \in X$ with $z' = e(x')$.

Then $z'_i = f_i(x') \in Y_i \setminus \overline{f_i(X \setminus U))} \subseteq Y_i \setminus f_i(X \setminus U)$.

It follows that $x' \in U$ (otherwise $f_i(x') \in f_i(X \setminus U)$, a contradiction).

Thus $z' \in e(U)$ showing that $e(U)$ is open in $e(X)$.

So $e : X \to e(X)$ is an open function and $e : X \to \prod_{i \in I} Y_i$ is an embedding.

□

Remark. If one of the functions of $\mathcal{A}$, say f_{i_0}, is already an embedding then $\mathcal{A}$ is point-separating and separating points from closed sets (because of f_{i_0}) and therefore $e : X \to \prod_{i \in I} Y_i$ is an embedding.

Example. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = x$ and $f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = \sin x$ then $\mathbb{R}$ can be embedded in $\mathbb{R}^2$ via $e(x) = (x, \sin x)$.

Observe that f_1 is already an embedding.

As an application of the diagonal lemma we are able to prove a metrization theorem.

For this let (X, τ) be a regular second countable space.

We mentioned earlier (without proof) that every regular Lindelöf space is normal, and so (X, τ) is a normal space.

Let $\mathcal{B}$ be a countable base for (X, τ) .

We consider pairs (B, B') with $B, B' \in \mathcal{B}$ and $\overline{B} \subseteq B'$.

For each such pair we have $\overline{B} \cap (X \setminus B') = \emptyset$. By the lemma of Urysohn there exists a continuous function

$$f_{(B,B')} : X \to [0, 1] \quad \text{with} \quad f_{(B,B')}|_{\overline{B}} = 1 \ \text{ and } \ f_{(B,B')}|_{X \setminus B'} = 0 \ .$$

Thus the family $\mathcal{A} = \{f_{(B,B')} : X \to [0, 1] \ : \ B, B' \in \mathcal{B} \ , \ \overline{B} \subseteq B'\}$ is a countable (!) family of continuous functions.

Let $x \neq y$. Since (X, τ) is regular there exist $B, B' \in \mathcal{B}$ such that $x \in \overline{B} \subseteq B'$ and $y \notin B'$.

Hence $f_{(B,B')}(x) = 1 \neq 0 = f_{(B,B')}(y)$ showing that $\mathcal{A}$ is a point-separating family.

Now let $A \subseteq X$ be closed and $x \notin A$. Then there exist $B, B' \in \mathcal{B}$ such that $x \in \overline{B} \subseteq B' \subseteq X \setminus A$ (since $X \setminus A$ is open).

Then $f_{(B,B')}(x) = 1$ and, since $A \subseteq X \setminus B'$,

$$f_{(B,B')}(A) \subseteq \{0\} \text{ , and so } \overline{f_{(B,B')}(A)} \subseteq \{0\} \quad \text{and}$$

$$f_{(B,B')}(x) \notin \overline{f_{(B,B')}(A)} \ .$$

Hence $\mathcal{A}$ separates points from closed sets.

By the diagonal lemma (X, τ) can be embedded in a countable product of $[0,1]$.

Next we point out that the countable product of metric spaces is again a metric space. Let (X_n, d_n) be a metric space for each $n \in \mathbb{N}$ (wlog $d_n \leq 1$) and let $X = \prod_{n\in\mathbb{N}} X_n$.

One can show that $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n)$ is a metric on X generating the product topology.

Therefore we have

Theorem. Let (X,τ) be regular and second countable. Then (X,τ) is metrizable.

Corollary.

1) Let (X,τ) be second countable. Then

(X,τ) is metrizable $\Leftrightarrow$ (X,τ) is regular

2) Let (X,τ) be compact and T_2 . Then

(X,τ) is metrizable $\Leftrightarrow$ (X,τ) is second countable.

Now suppose that (X,τ) is completely regular and let

$\mathcal{A} = \{f_i : X \to [0,1] \ : \ f_i \text{ is continuous }, \ i \in I\}$

the family of **all** continuous functions $X \to [0,1]$.

Then obviously the diagonal lemma is applicable and we obtain an embedding $e : X \to [0,1]^I$, and, by the theorem of Tychonoff, $[0,1]^I$ is a compact T_2-space.

Therefore

Theorem. (X,τ) is completely regular if and only if (X,τ) can be embedded in a compact T_2-space.

Definition. A (T_2-)**compactification** of a space $(X\tau)$ is a pair (Y, f) where (Y, σ) is a compact (T_2-)space and $f : X \to Y$ is an embedding with $\overline{f(X)} = Y$.

Remarks.

i) If $f : X \to Y$ is an embedding and Y is compact we obtain a compactification (Y^*, f) by setting $Y^* = \overline{f(X)}$.

ii) We already considered the 1-point-compactification of locally compact T_2-spaces and proved that every completely regular space has a T_2-compactification.

iii) $(0, 1) \to [0, 1]$ with the inclusion function is another compactification of $(0, 1)$.

iv) Intuitively, we "add" points to X to obtain a set Y . Then we ask if there is a suitable topology on Y such that X is a subspace of Y .

One of the most prominent compactifications in topology is the Stone-Cech-compactification.

Let (X, τ) be completely regular.

Consider $C^*(X) = \{f : X \to \mathbb{R} \ : \ f$ is continuous and bounded$\}$.

Obviously, for each $f \in C^*(X)$ there is a compact interval $I_f \subseteq \mathbb{R}$ such that $f(X) \subseteq I_f$ and so we can consider a function $f \in C^*(X)$ also as a function $f : X \to I_f$.

It is also obvious that $C^*(X)$ fulfills the requirements of the diagonal lemma and therefore

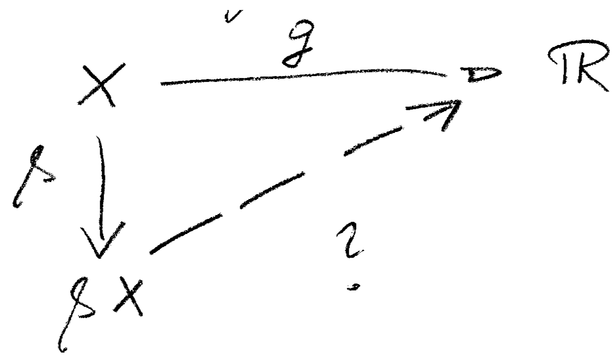
$$e : X \to \prod_{f \in C^*(X)} I_f \quad \text{with} \quad p_f \circ e = f \quad \text{is an embedding.}$$

We set $\beta X = \overline{e(X)}$ and $\beta : X \to \beta X$ by $\beta(x) = e(x)$.

By the theorem of Tychonoff $\prod_{f \in C^*(X)} I_f$ is a compact T_2-space, and as a closed subspace of $\prod_{f \in C^*(X)} I_f$, βX is also a compact T_2-space.

Definition. $(\beta X, \beta)$ is called the **Stone-Cech-compactification** of (X, τ) .

Now let $g \in C^*(X)$.



Theorem. There exists a continuous function $\widehat{g} : \beta X \to \mathbb{R}$ such that $\widehat{g} \circ \beta = g$.

(If we identify X with $\beta(X) \subseteq \beta X$ then every continuous and bounded function on X can be continuously extended to βX . We say that X is C^*-embedded in βX .)

Proof. Let $g(X) \subseteq I_g$ and $p_g : \prod_{f \in C^*(X)} I_f \to I_g$ the corresponding projection.

Let $\widehat{g} = p_g|_{\beta X} : \beta X \to I_g \subseteq \mathbb{R}$. Then $\widehat{g}$ is continuous.

Also, $\widehat{g} \circ \beta(x) = p_g|_{\beta X} \circ \beta(x) = p_g \circ e(x) = g(x)$.

Observe also, that $\widehat{g}$ is uniquely determined since any two extensions coincide on $e(X)$ and therefore on $\beta X = \overline{e(X)}$. $\square$

We now prove the so-called "characteristic property" of the Stone-Cech-compactification.

Theorem. Let (X, τ) be completely regular and (Y, σ) a compact T_2-space.

For every continuous function $f : X \to Y$ there is a continuous function $h : \beta X \to Y$ such that $h \circ \beta = f$.

(i.e. f can be continuously extended to βX.)

Proof. We first observe that if $k : Z_1 \to Z_2$ is a continuous and closed function then $k(\overline{A}) = \overline{k(A)}$ for each $A \subseteq Z_1$.

It follows from the previous discussions that there is also an embedding

$$\widehat{e} : Y \to \prod_{g \in C^*(Y)} I_g$$

where I_g is a compact interval of $\mathbb{R}$ with $g(Y) \subseteq I_g$ for each $g \in C^*(Y)$, and we have $p_g \circ \widehat{e} = g$.

Let $g \in C^*(Y)$. Then $g \circ f \in C^*(X)$ and $g \circ f(X) \subseteq I_g$.

By the previous theorem there exists a (unique) continuous function $\widehat{g \circ f} : \beta X \to \mathbb{R}$ such that $\widehat{g \circ f} \circ \beta = g \circ f$ and

$$\widehat{g \circ f}(\beta X) = \widehat{g \circ f}(\overline{\beta(X)}) = \overline{(\widehat{g \circ f} \circ \beta)(X)} = \overline{g \circ f(X)} \subseteq \overline{I_g} = I_g$$

i.e. $\widehat{g \circ f}$ maps βX into I_g.

Hence the function $\widetilde{h} : \beta X \to \prod_{g \in C^*(Y)} I_g$ where $p_g \circ \widetilde{h} = \widehat{g \circ f}$ is continuous.

We now claim that $\widetilde{h} \circ \beta = \widehat{e} \circ f : X \to \prod_{g \in C^*(Y)} I_g$.

For $x \in X$ and $g \in C^*(Y)$ we have

$$p_g(\widetilde{h} \circ \beta(x)) = (p_g \circ \widetilde{h})(\beta(x)) = \widehat{g \circ f}(\beta(x)) = \widehat{g \circ f} \circ \beta(x) = g \circ f(x) =$$

$$= p_g \circ \widehat{e}(f(x)) = p_g(\widehat{e} \circ f(x))$$

From this it follows that $\widetilde{h} \circ \beta(x) = \widehat{e} \circ f(x)$ for each $x \in X$ and so $\widetilde{h} \circ \beta = \widehat{e} \circ f$.

In addition we have

$$\widetilde{h}(\beta X) = \widetilde{h}(\overline{\beta(X)}) \subseteq \overline{\widetilde{h} \circ \beta(X)} = \overline{\widehat{e} \circ f(X)} = \overline{\widehat{e}(f(X))} \subseteq \overline{\widehat{e}(Y)} = \widehat{e}(Y)$$

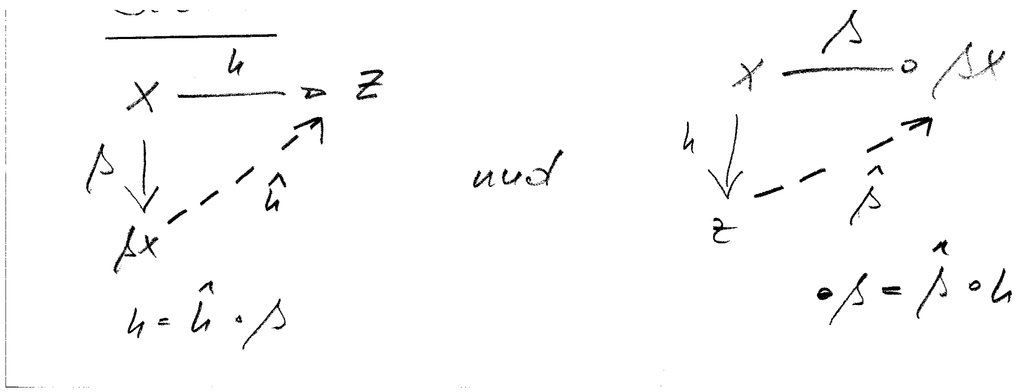
Since $\widehat{e} : Y \to \widehat{e}(Y)$ is a homeomorphism there is a continuous inverse function $\widehat{e}^{-1} : \widehat{e}(Y) \to Y$.

Then $h = \widehat{e}^{-1} \circ \widetilde{h} : \beta X \to Y$ is well defined and continuous and we have

$h \circ \beta = \widehat{e}^{-1} \circ \widetilde{h} \circ \beta = \widehat{e}^{-1} \circ \widehat{e} \circ f = f$. $\square$

Remark. βX is with respect to this extension property uniquely determined, i.e. if (Z, h) is another compactification of (X, τ) with this extension property then Z is homeomorphic to βX .

Proof.



Consider $\widehat{\beta} \circ \widehat{h} : \beta X \to \beta X$.

On the dense subset $\beta(X)$ we have

$\widehat{\beta} \circ \widehat{h}(\beta(x)) = \widehat{\beta}(h(x)) = \beta(x)$, i.e.

$\widehat{\beta} \circ \widehat{h}|_{\beta(X)} = \mathrm{id}_{\beta X}|_{\beta(X)}$ and so $\widehat{\beta} \circ \widehat{h} = \mathrm{id}_{\beta X}$

In the same manner we obtain $\widehat{h} \circ \widehat{\beta} = \mathrm{id}_Z$. Therefore βX and Z are homeomorphic. $\square$

We now ask for the "size" of βX resp. of $\beta X \setminus X$.

Clearly, if (X, τ) is already compact and T_2 then $\beta X = X$. In general, βX can be very large.

For this we consider $X = \mathbb{N}$ with the discrete topology.

We first mention (without proof) a very important result in topology.

Theorem. (E. Hewitt)

Let (X_i, τ_i) be separable for each $i \in I$, and $|I| \leq \mathrm{c}$.

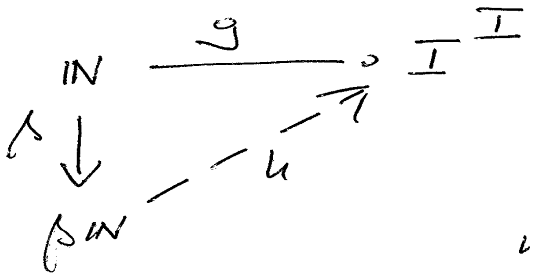
Then $\prod_{i \in I} X_i$ is separable.

Theorem. $|\beta\mathbb{N}| = \mathrm{c}^{\mathrm{c}} = (2^{\aleph_0})^{\mathrm{c}} = 2^{\mathrm{c}}$

Proof. If $I = [0,1]$, then by the previous theorem $I^I = \{x : I \to I\}$ is separable, so there exists a countable dense subset $D \subseteq I^I$.

Then there is an injective function $g : \mathbb{N} \to I^I$ with $g(\mathbb{N}) = D$.

Observe that I^I is compact and T_2 and, since $\mathbb{N}$ has the discrete topology, g is continuous.



It follows that there exists a continuous (and closed function) $h : \beta\mathbb{N} \to I^I$ such that $g = h \circ \beta$.

Then $h(\beta\mathbb{N}) = h(\overline{\beta(\mathbb{N})}) = \overline{h(\beta(\mathbb{N}))} = \overline{g(\mathbb{N})} = \overline{D} = I^I$.

Hence h is surjective and so $|\beta\mathbb{N}| \geq |I^I| = \mathrm{c}^{\mathrm{c}}$.

On the other hand we know that $\beta\mathbb{N} \subseteq \prod_{f \in C^*(\mathbb{N})} I_f$.

Since each I_f is homeomorphic to $I = [0,1]$, $\prod_{f \in C^*(\mathbb{N})} I_f$ is homeomorphic to $I^{C^*(\mathbb{N})}$.

Now $|I| = \mathrm{c}$ and $|C^*(\mathbb{N})| \leq |\mathbb{R}^{\mathbb{N}}| = \mathrm{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathrm{c}$.

Hence $|\beta\mathbb{N}| \leq |I^{C^*(\mathbb{N})}| \leq \mathrm{c}^{\mathrm{c}}$ and so $|\beta\mathbb{N}| = \mathrm{c}^{\mathrm{c}}$ resp. $|\beta\mathbb{N} \setminus \mathbb{N}| = \mathrm{c}^{\mathrm{c}}$. $\square$

In concluding we mention without proof some further results about $\beta\mathbb{N}$.

We may identify $\mathbb{N}$ with $\beta(\mathbb{N}) \subseteq \beta\mathbb{N}$ so that we consider $\mathbb{N}$ as a subset of $\beta\mathbb{N}$.

1) Let $h : \beta\mathbb{N} \to \beta\mathbb{N}$ be a homeomorphism.

Then h maps isolated points to isolated points. Since $\mathbb{N}$ is dense in $\beta\mathbb{N}$ the only isolated points of $\beta\mathbb{N}$ are the points of $\mathbb{N}$.

Therefore h induces a permutation of $\mathbb{N}$.

Conversely, each permutation of $\mathbb{N}$ induces a homeomorphism $\beta\mathbb{N} \to \beta\mathbb{N}$.

So there are $\mathrm{c} = 2^{\aleph_0}$ homeomorphisms $\beta\mathbb{N} \to \beta\mathbb{N}$.

2) Let $E \subseteq \beta\mathbb{N}$ be a countable subset.

Then $\overline{E}^{\beta\mathbb{N}}$ is homeomorphic to βE .

3) Let $F \subseteq \beta\mathbb{N}$ be infinite and closed.

Then F contains a subspace homeomorphic to $\beta\mathbb{N}$.

In particular, $\beta\mathbb{N} \setminus \mathbb{N}$ contains a subspace homeomorphic to $\beta\mathbb{N}$.

4) Let $\mathbb{N}_1$ be the set of even integers and $\mathbb{N}_2$ be the set of odd integers. Obviously $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$.

Then $\beta\mathbb{N} = \overline{\beta\mathbb{N}_1} \cup \overline{\beta\mathbb{N}_2}$ and $\overline{\beta\mathbb{N}_1} \cap \overline{\beta\mathbb{N}_2} = \emptyset$.

Observe that both $\overline{\beta\mathbb{N}_1}$ and $\overline{\beta\mathbb{N}_1}$ are homeomorphic to $\beta\mathbb{N}$.

5) Consider $\mathbb{N} \cup \{p \ : \ p$ is a free ultrafilter on $\mathbb{N}\}$.

On this set one can define a suitable topology so that the resulting space is homeomorphic to $\beta\mathbb{N}$.