Topology - Exercise Sheet 1

1. Show that, in general, in a topological space the arbitrary union of closed sets need not be closed.

   Definition. A family $\{A_i \,:\, i \in I\}$ of subsets of a space (X, τ) is called **locally finite**, if each point $x \in X$ has a neighbourhood U_x that intersects at most finitely many A_i .

   Show that if $\{A_i \,:\, i \in I\}$ is locally finite then $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$.
   (In particular, $\bigcup_{i \in I} A_i$ is closed whenever all A_i are closed.)

2. A subset $U \subseteq X$ is called **regular open** (resp. **regular closed**) if $U = \mathrm{int}\overline{U}$ (resp. $U = \overline{\mathrm{int}U}$). Show the following:
   (i) $U \subseteq X$ is regular open $\Leftrightarrow$ $\exists\, A \subseteq X$ closed with $U = \mathrm{int}A$.
   (ii) $U \subseteq X$ is regular open $\Leftrightarrow$ $X \setminus U$ is regular closed.
   (iii) $U, V \subseteq X$ regular open $\Rightarrow$ $U \cap V$ is regular open.
   (iv) Find a subset of $\mathbb{R}$ that is open but not regular open.

3. Let X be a set and Φ be a function $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ with the following properties:
   $\Phi(\emptyset) = \emptyset$, $A \subseteq \Phi(A)$, $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ and $\Phi(\Phi(A)) = \Phi(A)$.
   Show that there is a unique topology τ on X such that $\Phi(A) = \overline{A}$ for each $A \subseteq X$.
   (Φ is called a Kuratowski closure-operator)

4. Let (X, τ) be a space and $B \subseteq A \subseteq X$.
   Show that $U \subseteq A$ is a neighbourhood of $x \in A$ with respect to $(A, \tau|_A)$ if and only if there is a neighbourhood $V \subseteq X$ of x with respect to (X, τ) with $U = V \cap A$. Furthermore, $\tau|_B = (\tau|_A)|_B$.

5. Let (X, τ) be a space and $A \subseteq X$.
   (i) $B \subseteq A$ is closed in A $\Leftrightarrow$

there exists $F \subseteq X$ closed in X such that $B = F \cap A$.

(ii) For $B \subseteq A$, the closure of B with respect to $(A, \tau|_A)$ is denoted by $\overline{B}^A$. Then $\overline{B}^A = \overline{B} \cap A$.

6. Let τ be the cofinite topology on a set X . Show:
   i) If X is countable then (X, τ) is second countable.
   ii) If X is uncountable then (X, τ) is not first countable.

7. Show that every subspace of a second countable (resp. a first countable) space is second countable (resp. first countable). From this conclude that $\mathbb{R}$ cannot have an uncountable discrete subspace and that the Niemitzky plane is not second countable.

8. Show that the Sorgenfrey line is not second countable.

9. **Definition.** Let (X, τ) be a space. We say that a sequence (x_n) converges to $x \in X$, $x_n \to x$, if $\forall\, U \in \mathcal{U}(x)$ $\exists\, N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$.

   Now let (X, τ) be first countable. Show that
   (i) $x \in \overline{A}$ $\Leftrightarrow$ $\exists\, (a_n) \subseteq A$ such that $a_n \to x$.
   (ii) $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for every sequence (x_n) with $x_n \to x_0$ it holds that $f(x_n) \to f(x_0)$.