Topology - Exercise Sheet 2

1. Show that a space is T_3 if and only if for each $x \in X$ the closed neighbourhoods of x form a neighbourhood base (i.e. each neighbourhood contains a closed neighbourhood).

2. Show that the arbitrary product of T_3-spaces is a T_3-space.

3. For each $i \in I$ let (X_i, τ_i) be a non-trivial (i.e. $|X_i| > 1$) first countable space. Let $X = \prod_{i \in I} X_i$ be the product space with the product topology τ .
Show that (X, τ) is first countable if $|I|$ is at most countable and that (X, τ) is not first countable if $|I|$ is uncountable.

4. Let X be a set. The set $\{f : X \to \mathbb{R}\}$ can be written as $\mathbb{R}^X$ (all factors are $\mathbb{R}$ and the index set is X) and we can consider the product topology on this set.
Show that $(f_n) \to f$ if and only if for each $x \in X$ we have $f_n(x) \to f(x)$.

5. Prove with the Lemma of Zorn that for each filter $\mathcal{F}$ on a set X there exists an ultrafilter $\mathcal{U}$ satisfying $\mathcal{F} \subseteq \mathcal{U}$.

6. Show that every subspace of the Sorgenfrey line is Lindelöf.
(Hint: Let $\{O_i \ : \ i \in I\}$ be a family of sets open in the Sorgenfrey line. For each $i \in I$ let $\text{int}O_i$ be the interior of O_i with respect to the usual topology on $\mathbb{R}$. Prove that $A = \bigcup_{i \in I} O_i \setminus \bigcup_{i \in I} \text{int}O_i$ is at most countable. Use also the facts that $\mathbb{R}$ with the usual topology is second countable and that every family of pairwise disjoint nonempty open sets must be countable.)

7. Let (X, τ) be the Sorgenfrey line. Show that $X \times X$ is not Lindelöf.

8. Let (X, τ) be the Sorgenfrey line and let $C \subseteq X$ be compact. Show that C is at most countable.
(Hint: For each $x \in C$ consider the open cover $C \subseteq [x, \infty) \cup \bigcup_{n \in \mathbb{N}} (-\infty, x - \frac{1}{n})$. As a consequence there is a nonempty euclidean open interval I_x such that $I_x \cap C = \emptyset$. Show that this family $\{I_x \ : \ x \in C\}$ of intervals is pairwise disjoint and therefore must be countable.)