Properties of $T_{1/4}$ spaces *

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Abstract

In 1997, Arenas, Dontchev and Ganster [1] introduced the class of $T_{1/4}$ spaces in their study of generalized continuity and λ-closed sets. The aim of this paper is to continue the investigation of $T_{1/4}$ spaces, in particular to consider products of $T_{1/4}$ spaces. As a result, we obtain an easy way to provide examples of $T_{1/4}$ spaces that fail to be $T_{1/2}$.

1 Introduction and Preliminaries

In 1997, Arenas, Dontchev and Ganster [1] introduced in their study of generalized continuity the notion of a λ-closed set in a topological space. A subset A of a space (X, τ) is called λ-closed if $A = L \cap F$, where L is a Λ-set, i.e. L is an intersection of open sets, and F is closed. Using λ-closed sets, the authors in [1] characterized T_0 spaces as those spaces where each singleton is λ-closed, and $T_{1/2}$ spaces as those spaces where every subset is λ-closed. The notion of a $T_{1/2}$ space has been introduced by Levine in [5]. Dunham [3] showed that a space (X, τ) is $T_{1/2}$ if and only if each singleton is open or closed. One of the most important examples of $T_{1/2}$ spaces is the digital line or Khalimsky line $(\mathbb{Z}, \kappa)$ (see e.g. [4]). The digital line is the set of integers $\mathbb{Z}$ with the topology κ having $\mathcal{S} = \{\{2m-1, 2m, 2m+1\} : m \in \mathbb{Z}\}$ as a subbase. Clearly, $(\mathbb{Z}, \kappa)$ fails to be T_1. However, each singleton of the form $\{2m\}$ is

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closed and each singleton of the form $\{2m-1\}$ is open. It should be observed that $(\mathbb{Z}, \kappa)$ is even a $T_{3/4}$ space (see [2]).

In [1], the authors also introduced a new separation axiom, called $T_{1/4}$. They pointed out that the class of $T_{1/4}$ spaces is strictly placed between the class of T_0 spaces and the class of $T_{1/2}$ spaces, and that $T_{1/4}$ spaces are precisely those spaces where each finite set is λ-closed. The purpose of this paper is to continue the study of $T_{1/4}$ spaces, in particular to investigate their behaviour under forming products. As a result, we obtain easy ways to construct non-$T_{1/4}$ spaces and also $T_{1/4}$ spaces that fail to be $T_{1/2}$.

Throughout this paper we do not assume any separation axioms unless stated explicitly. The interior and the closure of a subset A of a space (X, τ) will be denoted by $\mathrm{int}A$ and $\overline{A}$, respectively.

2 Properties of $T_{1/4}$-spaces

Definition 1 [1] A space (X, τ) is called a $T_{1/4}$ space if for every finite subset $F \subseteq X$ and every point $y \notin F$ there exists a subset $A \subseteq X$ such that $F \subseteq A$, $y \notin A$ and A is open or closed.

It is obvious that every $T_{1/4}$ space is T_0 and that (X, τ) is $T_{1/2}$ if and only if for every subset $F \subseteq X$ and every point $y \notin F$ there exists a subset $A \subseteq X$ such that $F \subseteq A$, $y \notin A$ and A is open or closed. The following observation is easily verified.

Proposition 2.1 Let (X, τ) be a $T_{1/4}$ space. Then every subspace of X is a $T_{1/4}$ space.

Recall that a subset F of a space (X, τ) is called locally finite if every point $x \in X$ has an open neighbourhood U_x such that $F \cap U_x$ is at most finite.

Theorem 2.2 For a space (X, τ) the following are equivalent:

1) (X, τ) is a $T_{1/4}$ space,

2) For every locally finite subset $F \subseteq X$ and every point $y \notin F$ there exists a subset $A \subseteq X$ such that $F \subseteq A$, $y \notin A$ and A is open or closed.

Proof. 1) $\Rightarrow$ 2) : Let $F \subseteq X$ be locally finite and let $y \notin F$. If F is finite, we are done. So let us assume that F is infinite. If y has an open neighbourhood U with empty intersection with F then $A = X \setminus U$ is the required set. Otherwise $y \in \overline{F}$ and, since F is locally finite, there exists an open neighbourhood U of y such that $U \cap F$ is finite, say $\{x_1, ..., x_k\}$, and $y \in \overline{\{x_i\}}$ for some x_i. Now pick any $x \in F \setminus \{x_1, ..., x_k\}$. Then $\{x, x_1, ..., x_k\}$ is a finite set not containing y. Since (X, τ) is $T_{1/4}$ there must be a subset $A_x \subseteq X$ such that $\{x, x_1, ..., x_k\} \subseteq A_x$, $y \notin A_x$ and A_x is open or closed. Since $y \in \overline{\{x_i\}} \subseteq \overline{A_x}$, A_x cannot be closed, thus A_x must be open. If $A = \bigcup\{A_x \ : \ x \in F \setminus \{x_1, ..., x_k\}\}$ then A is an open set containing F such that $y \notin A$, and we are done.
2) $\Rightarrow$ 1) : This is clear since every finite subset is a locally finite subset. □

Remark 2.3 It is natural to consider the following variation of $T_{1/4}$ spaces. A space (X, τ) is said to be $T^c_{1/4}$ if for every at most countable subset $F \subseteq X$ and every point $y \notin F$ there exists a subset $A \subseteq X$ such that $F \subseteq A$, $y \notin A$ and A is open or closed. Clearly, every $T_{1/2}$ space is $T^c_{1/4}$ and every $T^c_{1/4}$ space is $T_{1/4}$. None of these implications is reversible, however, as we shall see in our next two examples.

Example 2.4 Let X be the set of nonnegative integers with the topology τ where $U \in \tau$ if and only if $U = \emptyset$ or $0 \in U$ and $X \setminus U$ is finite. It has been shown in [1] Example 3.2, that (X, τ) is $T_{1/4}$. Now let $F = X \setminus \{0\}$. Then F is countable and $0 \notin F$. Since $\{0\}$ is neither open nor closed, (X, τ) cannot be $T^c_{1/4}$.

Example 2.5 Let Y be an uncountable set, $p \notin Y$ and let $X = Y \cup \{p\}$. A topology τ on X is defined by declaring a subset $U \subseteq X$ to be open if $U = \emptyset$ or $p \in U$ and $X \setminus U$ is at most countable. Since $\{p\}$ is neither open nor closed, (X, τ) cannot be $T_{1/2}$. Now let $F \subseteq X$ be countable and let $y \notin F$. If $y \neq p$ then $\{y\}$ is closed and so $F \subset X \setminus \{y\} = A$ which is open. If $y = p$ then F is closed and we are also done in showing that (X, τ) is $T^c_{1/4}$.

3 Products of $T_{1/4}$-spaces

Theorem 3.1 Let (X,τ) and (Y,σ) be topological spaces such that $X \times Y$ is $T_{1/4}$. Then both spaces (X,τ) and (Y,σ) are $T_{1/4}$ and at least one of the spaces must be T_1.

Proof. Since (X,τ) and (Y,σ) are homeomorphic to subspaces of $X \times Y$, it follows from Proposition 2.1 that both spaces have to be $T_{1/4}$.
Now suppose that neither (X,τ) nor (Y,σ) is T_1. Then there exist $x, x_1 \in X$, $x \neq x_1$ such that $x_1 \in \overline{\{x\}}$, and $y, y_1 \in Y$, $y \neq y_1$ such that $y_1 \in \overline{\{y\}}$. Let $F = \{(x,y_1),(x,y),(x_1,y_1)\} \subseteq X \times Y$. Then $(x_1,y) \notin F$. By hypothesis, there exists a set $A \subseteq X \times Y$ such that $F \subseteq A$, $(x_1,y) \notin A$ and A is closed or open.
If A is closed, i.e. if $X \times Y \setminus A$ is open, there exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $x_1 \in U$, $y \in V$ and $(U_1 \times V) \cap F = \emptyset$. However, since $x_1 \in \overline{\{x\}}$, we have $x \in U_1$ and so $(x,y) \in U_1 \times V$, a contradiction. If A is open, then there exist open sets $U_1 \subseteq X$ and $V_1 \subseteq Y$ such that $x_1 \in U_1$, $y_1 \in V_1$ and $U_1 \times V_1 \subseteq A$. Since $y_1 \in \overline{\{y\}}$ we have $y \in V_1$ and thus $(x_1,y) \in A$, a contradiction. As a consequence, at least one of the spaces (X,τ) or (Y,σ) has to be T_1 . □

Theorem 3.2 Let $X = \prod_{i \in I} X_i$ be the product of topological spaces (X_i,τ_i) , $i \in I$. If X is $T_{1/4}$ then all spaces (X_i,τ_i) are $T_{1/4}$ and at most one factor space is not T_1.

Proof. Each (X_i,τ_i) is homeomorphic to a subspace of X, hence it is $T_{1/4}$ by Proposition 2.1. Now suppose that (X_i,τ_i) and (X_j,τ_j) , $i \neq j$, are not T_1 . Since $X_i \times X_j$ is homeomorphic to a subspace of X, it has to be $T_{1/4}$ by Proposition 2.1. However, this is a contradiction to Theorem 3.1. □

Using our previous results it is easy to provide examples of T_0 spaces that are not $T_{1/4}$. For example we have

Corollary 3.3 The digital plane $\mathbb{Z}^2$, i.e. the product of two copies of the digital line $(\mathbb{Z},\kappa)$, fails to be $T_{1/4}$.

We now address the question of when a product of topological spaces is $T_{1/4}$. As we shall see, the converses of Theorem 3.1 and Theorem 3.2 also hold.

Theorem 3.4 Let (X, τ) be $T_{1/4}$ and let (Y, σ) be T_1 . Then $X \times Y$ is $T_{1/4}$.

Proof. Let $F = \{(x_1, y_1), ..., (x_n, y_n)\} \subseteq X \times Y$ and let $(x_0, y_0) \notin F$. We denote the natural projections by $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. Let $F' = \{(x_i, y_i) \in F : y_i = y_0\}$ and $F'' = \{(x_i, y_i) \in F : y_i \neq y_0\}$. Then $F = F' \cup F''$, $x_0 \notin p_1(F')$ and $y_0 \notin p_2(F'')$. Since $p_1(F') \subseteq X$ is finite and (X, τ) is $T_{1/4}$, there is a subset $A \subseteq X$ such that $p_1(F') \subseteq A$, $x_0 \notin A$ and A is open or closed.
If A is closed in (X, τ), then $A \times \{y_0\}$ is closed in $X \times Y$ such that $F' \subseteq A \times \{y_0\}$ and $(x_0, y_0) \notin A \times \{y_0\}$. Also, $X \times p_2(F'')$ is closed in $X \times Y$, $F'' \subseteq X \times p_2(F'')$ and $(x_0, y_0) \notin X \times p_2(F'')$. Hence there exists a closed set $B \subseteq X \times Y$, namely $B = (A \times \{y_0\}) \cup (X \times p_2(F''))$, such that $F \subseteq B$ and $(x_0, y_0) \notin B$.
If A is open in (X, τ), then $A \times Y$ is open in $X \times Y$ such that $F' \subseteq A \times Y$ and $(x_0, y_0) \notin A \times Y$. Also, $X \times (Y \setminus \{y_0\})$ is open in $X \times Y$, $F'' \subseteq X \times (Y \setminus \{y_0\})$ and $(x_0, y_0) \notin X \times (Y \setminus \{y_0\})$. So there is an open set $B \subseteq X \times Y$, namely $B = (A \times Y) \cup (X \times (Y \setminus \{y_0\}))$, such that $F \subseteq B$ and $(x_0, y_0) \notin B$. This proves that $X \times Y$ is $T_{1/4}$. □

Theorem 3.5 Let $X = \prod_{i \in I} X_i$ be the product of topological spaces (X_i, τ_i) , $i \in I$. If (X_i, τ_i) is $T_{1/4}$ for some $i \in I$ and all other spaces (X_j, τ_j) , $j \neq i$, are T_1 , then X is $T_{1/4}$.

Proof. Clearly X is homeomorphic to $X_i \times (\prod_{j \neq i} X_j)$. By Theorem 3.4, it follows that X is $T_{1/4}$, since $\prod_{j \neq i} X_j$ is T_1 . □

The following result is due to Dunham [3].

Theorem 3.6 ([3], Theorem 4.6) The product $X \times Y$ of the spaces (X, τ) and (Y, σ) is $T_{1/2}$ if and only if one of the following conditions holds:

(a) Both spaces (X, τ) and (Y, σ) are T_1 .

(b) One of the spaces is $T_{1/2}$ but not T_1, while the other is discrete.

Remark 3.7 Combining Theorem 3.4 and Theorem 3.6 provides a way to give a variety of examples of $T_{1/4}$ spaces that fail to be $T_{1/2}$. In particular, if (X, τ) is $T_{1/2}$ (and thus $T_{1/4}$) but not T_1 and (Y, σ) is a non-discrete T_1 space then $X \times Y$ is $T_{1/4}$ by Theorem 3.4, but not $T_{1/2}$ by Theorem 3.6.

Example 3.8 If $(\mathbb{Z}, \kappa)$ denotes the digital line and $\mathbb{R}$ the usual space of reals, then $\mathbb{Z} \times \mathbb{R}$ is, by Remark 3.7, a $T_{1/4}$ space but not $T_{1/2}$. The space $\mathbb{Z} \times \mathbb{R}$ is interesting insofar as we may consider maps $H : \mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ which describe the movement of digital pictures, i.e. subsets of $\mathbb{Z}$, through time.

References

[1] F. G. Arenas, J. Dontchev and M. Ganster, *On λ-sets and the dual of generalized continuity* , Q & A in General Topology 15 (1997), 3-13.

[2] J. Dontchev and M. Ganster, *On δ-generalized closed sets and $T_{3/4}$-spaces* , Mem. Fac. Sci. Kochi Univ. Ser. A Math. 17 (1996), 15-31.

[3] W. Dunham, *$T_{1/2}$-spaces* , Kyungpook Math. J. 17 (1977), 161-169.

[4] E. D. Khalimsky, R. Kopperman and P.R. Meyer, *Computer graphics and connected topologies on finite ordered sets* , Topology Appl. 36 (1990), 1-17.

[5] N. Levine, *Generalized closed sets in topology* , Rend. Circ. Mat. Palermo 19 (2) (1970), 89-96.

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