On $b\tau$-closed sets

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Abstract. This paper is closely related to the work of Cao, Greenwood and Reilly in [10] as it expands and completes their fundamental diagram by considering b-closed sets. In addition, we correct a wrong assertion in [10] about $T_{\beta s}$-spaces.

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1. Introduction and Preliminaries

In recent years quite a number of generalizations of closed sets has been considered in the literature. We recall the following definitions:

Definition 1.1. *Let (X, τ) be a topological space. A subset $A \subseteq X$ is called*

(1) α-closed [18] *if* $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subseteq A$,
(2) semi-closed [15] *if* $\mathrm{int}(\mathrm{cl}(A)) \subseteq A$,
(3) preclosed [17] *if* $\mathrm{cl}(\mathrm{int}(A)) \subseteq A$,
(4) b-closed [3] *if* $\mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{cl}(\mathrm{int}(A)) \subseteq A$,
(5) semi-preclosed [2] *or* β-closed [1] *if* $\mathrm{int}(\mathrm{cl}(\mathrm{int}(A))) \subseteq A$.

The complement of an α-closed (resp. semi-closed, preclosed, b-closed, β-closed) set is called *α-open* (resp. *semi-open*, *preopen*, *b-open*, *β-open*). The smallest α-closed (resp. semi-closed, preclosed, b-closed, β-closed) set containing $A \subseteq X$ is called the *α-closure* (resp. *semi-closure*, *preclosure*, *b-closure*, *β-closure*) of A and shall be denoted by $\mathrm{cl}_\alpha(A)$ (resp. $\mathrm{cl}_s(A)$, $\mathrm{cl}_p(A)$, $\mathrm{cl}_b(A)$, $\mathrm{cl}_\beta(A)$).

In 2001, Cao, Greenwood and Reilly [10] introduced the concept of qr-closed sets to deal with various notions of generalized closed sets that had been considered in the literature so far. If $\mathcal{P} = \{\tau, \alpha, s, p, \beta\}$ and $q, r \in \mathcal{P}$ then a subset $A \subseteq X$ is called *qr-closed* if $\mathrm{cl}_q(A) \subseteq U$ whenever $A \subseteq U$ and U is r-open. (For convenience we denote $\mathrm{cl}(A)$ by $\mathrm{cl}_\tau(A)$ and open (resp. semi-open, preopen) by τ-open (resp. s-open, p-open).)

In the following we shall consider the expanded family $\mathcal{P} \cup \{b\}$. As in Corollary 2.6 of [10], it is easily established that the concept of a $b\tau$-closed set yields the only new type of sets that can be gained by utilizing the b-closure (resp. the b-interior) in the context of qr-closed sets. Thus we give

Definition 1.2. *Let (X, τ) be a topological space. A subset $A \subseteq X$ is called* $b\tau$-closed *if* $\mathrm{cl}_b(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open. The complement of a $b\tau$-closed set is called* $b\tau$-open.

Remark 1.3. The concepts of ss-closed (resp. $s\tau$-closed, $p\tau$-closed, $\beta\tau$-closed) sets have been first introduced in the literature under the name of sg-closed [5] (resp. gs-closed [4], gp-closed [16], gsp-closed [11]) sets.

We also consider the following classes of topological spaces:

Definition 1.4. *A topological space* (X, τ) *is called*

(1) sg-submaximal *if every codense subset of* (X, τ) *is ss-closed,*
(2) T_{gs} *if every* $s\tau$*-closed subset of* (X, τ) *is ss-closed,*
(3) extremally disconnected *if the closure of each open subset of* (X, τ) *is open,*
(4) resolvable *if* (X, τ) *is the union of two disjoint dense subsets.*

For undefined concepts we refer the reader to [10] and [9] and the references given there.

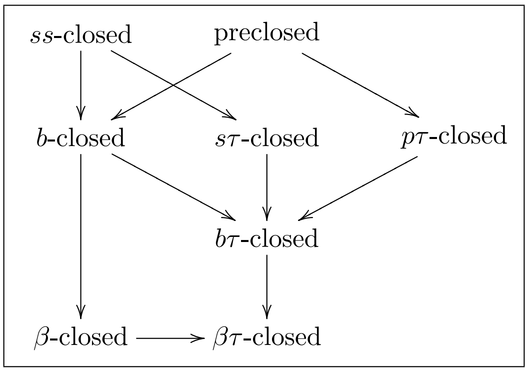
2. $b\tau$-closed sets and their relationships

In [10] the relationships between various types of generalized closed sets have been summarized in a diagram. We shall expand this diagram by adding b-closed sets and $b\tau$-closed sets.

Proposition 2.1. *Every ss-closed set in a topological space* (X, τ) *is b-closed.*

Proof. Let $A \subseteq X$ be ss-closed and let $x \in \mathrm{cl}_b(A)$. Since singletons are either preopen or nowhere dense (see [14]), we distinguish two cases. If $\{x\}$ is preopen, it is also b-open and hence $\{x\} \cap A \neq \varnothing$, i.e. $x \in A$. If $\{x\}$ is nowhere dense, then $X \setminus \{x\}$ is semi-open. Suppose that $x \notin A$. Then $A \subseteq X \setminus \{x\}$ and, since A is ss-closed, we have $\mathrm{cl}_b(A) \subseteq \mathrm{cl}_s(A) \subseteq X \setminus \{x\}$. Hence $x \notin \mathrm{cl}_b(A)$, a contradiction. Therefore $\mathrm{cl}_b(A) \subseteq A$, and so A is b-closed. □

The remaining relationships in the following diagram can easily be established.



We now address the question of when the above implications can be reversed.

Proposition 2.2. *Let (X,τ) be a topological space. Then:*

(1) *Each b-closed set is ss-closed iff (X,τ) is sg-submaximal.*
(2) *Each b-closed set is $s\tau$-closed iff (X,τ) is sg-submaximal.*
(3) *Each $s\tau$-closed set is b-closed iff (X,τ) is T_{gs}.*
(4) *Each $b\tau$-closed set is b-closed iff (X,τ) is T_{gs}.*
(5) *Each $b\tau$-closed set is $s\tau$-closed iff (X,τ) is sg-submaximal.*
(6) *Each $b\tau$-closed set is $p\tau$-closed iff (X,τ) is extremally disconnected.*
(7) *Each $b\tau$-closed set is β-closed iff (X,τ) is T_{gs}.*
(8) *Each β-closed set is b-closed iff* $\mathrm{cl}(W)$ *is open for every open resolvable subspace W of (X,τ).*

Proof. We will only show (1). The other assertions can be proved in a similar manner using the standard methods that can be found in [10], and (8) has been shown in [13].

First recall that a space is sg-submaximal iff every preclosed set is ss-closed (see [7]). If every b-closed set is ss-closed then every preclosed set ss-closed, i.e. (X,τ) is sg-submaximal.

Conversely, suppose that (X,τ) is sg-submaximal and let A be b-closed. Then A is the intersection of a semi-closed and a preclosed set (see [3]). Since every semi-closed set is ss-closed, by hypothesis, A is the intersection of two ss-closed sets. Since the arbitrary intersection of ss-closed sets is always ss-closed (see [12]), we conclude that A is ss-closed. □

Proposition 2.3. *Let (X,τ) be a topological space. Then the following statements are equivalent:*

(1) *Each $\beta\tau$-closed set is $b\tau$-closed.*
(2) *Each β-closed set is $b\tau$-closed.*

Proof. The necessity is clear, so we only have to show the sufficiency. Let A be a $\beta\tau$-closed set and U be an open subset of X such that $A \subseteq U$. Since A is $\beta\tau$-closed we have $\mathrm{cl}_\beta(A) \subseteq U$. Now, $\mathrm{cl}_\beta(A)$ is β-closed and hence $b\tau$-closed by hypothesis. Therfore $\mathrm{cl}_b(A) \subseteq \mathrm{cl}_b(\mathrm{cl}_\beta(A)) \subseteq U$ and thus our claim is proved. □

Remark 2.4. If $A \subseteq X$, then the largest b-open subset of A is called the *b-interior* of A and is denoted by $\mathrm{bint}(A)$. It is well known that $\mathrm{bint}(A) = (\mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))) \cap A$ (see [3]). Consequently, a subset A is $b\tau$-open iff for every closed subset F satisfying $F \subseteq A$ we have $F \subseteq \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$.

We shall now present one of our major results.

Theorem 2.5. *Let (X,τ) be a topological space. Then the following are equivalent:*

(1) *Each β-closed set is b-closed.*
(2) *Each β-closed set is $b\tau$-closed.*
(3) $\mathrm{cl}(W)$ *is open for every open resolvable subspace W of (X,τ).*

Proof. It is obvious that (1) $\Rightarrow$ (2). Furthermore, it has been shown in [13] that (3) $\Leftrightarrow$ (1), so we only have to prove that (2) $\Rightarrow$ (3).

If $W = \varnothing$, we are done, so let W be a nonempty open resolvable subspace and let E_1 and E_2 be disjoint dense subsets of $(W, \tau|_W)$. Suppose that there exists a point $x \in \mathrm{cl}(W) \setminus \mathrm{int}(\mathrm{cl}(W))$. Let $S = E_1 \cup \mathrm{cl}(\{x\})$. It is easily checked that $\mathrm{int}(S) = \varnothing$, $\mathrm{cl}(S) = \mathrm{cl}(W)$ and that S is β-open. By hypothesis, S is $b\tau$-open and so, since $\mathrm{cl}(\{x\}) \subseteq S$, we conclude that $\{x\} \subseteq \mathrm{cl}(\{x\}) \subseteq \mathrm{int}(\mathrm{cl}(S)) = \mathrm{int}(\mathrm{cl}(W))$. This is, however, a contradiction to our assumption and so $\mathrm{cl}(W)$ has to be open. □

3. A Remark on $T_{\beta s}$-spaces

In [10] a space has been called $T_{\beta s}$ if every $\beta\tau$-closed subset of (X, τ) is $s\tau$-closed. We observe that this is equivalent to the property that every β-closed subset is $s\tau$-closed, see [6]. Using some of our previous results, we are now able to give the following characterization.

Theorem 3.1. *Let (X, τ) be a topological space. Then the following are equivalent:*

(1) *(X, τ) is a $T_{\beta s}$-space.*
(2) *Every β-closed set is b-closed and (X, τ) is sg-submaximal.*
(3) *Every β-closed set is ss-closed.*

Note that the last property in the above theorem has been fully characterized in [8]. It was shown there that every β-closed set is ss-closed iff (X, τ^α) is g-submaximal, where τ^α denotes the α-topology of (X, τ) (see [18]), and a space is called g-submaximal if each codense set is $\tau\tau$-closed. So the claim in [10] that there exists a $T_{\beta s}$-space whose α-topology is not g-submaximal turns out to be wrong now. In fact, one can easily check that Example 3.5 of [10] is false. So $T_{\beta s}$ is not a new topological property as we have just seen.

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