On some questions about b-open sets *

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Abstract

In [2], D. Andrijevic observed that for every topological space (X, τ) we have $PO(X,\tau) \cup SO(X,\tau) \subseteq BO(X,\tau) \subseteq SPO(X,\tau)$, but that none of these inclusions can be replaced by equality. The purpose of this note is to characterize those spaces (X,τ) where $BO(X,\tau) = SPO(X,\tau)$ holds, and those spaces (X,τ) where $PO(X,\tau) \cup SO(X,\tau) = BO(X,\tau)$ holds.

1 Introduction and Preliminaries

Over the years quite a number of generalizations of the class of open sets in a topological space have been considered and widely investigated. These variations have many useful applications, e.g. they are utilized to provide results about decompositions of continuity. Among the most important classes of generalized open sets are the following:

Definition 1 A subset S of a topological space (X, τ) is called

(i) semi-open [9], if $S \subseteq \overline{\mathrm{int} S}$,

(ii) preopen [10], if $S \subseteq \mathrm{int}\overline{S}$,

(iii) semi-preopen [3] or β-open [1], if $S \subseteq \overline{\mathrm{int}\overline{S}}$,

(iv) b-open [2], if $S \subseteq \overline{\mathrm{int} S} \cup \mathrm{int}\overline{S}$.

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The classes of semi-open, preopen, semi-preopen and b-open subsets of a space (X, τ) are usually denoted by $SO(X, \tau)$, $PO(X, \tau)$, $SPO(X, \tau)$ and $BO(X, \tau)$ respectively. The class of b-open sets has been introduced and investigated in 1996 by Andrijevic [2]. He made the following fundamental observation.

Proposition 1.1 For every space (X, τ), $PO(X, \tau) \cup SO(X, \tau) \subseteq BO(X, \tau) \subseteq SPO(X, \tau)$ holds but none of these implications can be reversed.

The aim of this paper is to provide complete answers to the following open questions.
Question 1. For which spaces (X, τ) does $BO(X, \tau) = SPO(X, \tau)$ hold?
Question 2. For which spaces (X, τ) does $PO(X, \tau) \cup SO(X, \tau) = BO(X, \tau)$ hold?

Recall that a space (X, τ) is said to be resolvable if it has two disjoint dense subsets, otherwise it is called irresolvable. Every space (X, τ) can be represented uniquely as the disjoint union of a closed and resolvable subspace F and an open, hereditarily irresolvable subspace G (see e.g. [7] and [5]). We shall call this decomposition the Hewitt-representation of (X, τ). A space (X, τ) is called strongly irresolvable if every open subspace of (X, τ) is irresolvable. Clearly, if $X = F \cup G$ denotes the Hewitt-representation of (X, τ) then (X, τ) is strongly irresolvable if and only if $\overline{G} = X$.

A space (X, τ) is called extremally disconnected if $\overline{U}$ is open for every open subset U of (X, τ). Following E. van Douwen [4], $x \in X$ is said to be an $e.d.$-point of (X, τ) if $x \in \overline{U}$ implies $x \in \mathrm{int}\overline{U}$ for every open subset U of (X, τ). We shall denote the set of $e.d.$-points of a space (X, τ) by $ED(X, \tau)$. Clearly, a space (X, τ) is extremally disconnected if and only if $ED(X, \tau) = X$.

For the convenience of the reader we now list some known results that will be used later in our paper.

Proposition 1.2 Let (X, τ) be a space.

(i) If $U \subseteq X$ is open and $S \subseteq X$ is semi-open (preopen, semi-preopen, b-open respectively) then $U \cap S$ is semi-open (preopen, semi-preopen, b-open respectively),

(ii) for every subset $A \subseteq X$, $A \cap \mathrm{int}\overline{A}$ is preopen,

(iii) [2] $S \subseteq X$ is b-open if and only if S is the union of a semi-open set and a preopen set.

Proposition 1.3 For a space (X, τ) the following are equivalent:

(1) (X, τ) is extremally disconnected,

(2) $SO(X, \tau) \subseteq PO(X, \tau)$,

(3) $SPO(X, \tau) = PO(X, \tau)$,

(4) $BO(X, \tau) = PO(X, \tau)$.

Proof. (1) $\Leftrightarrow$ (2) has been shown in [8] and (1) $\Leftrightarrow$ (3) has been shown in [6]. Clearly, (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) follow immediately from Proposition 1.1.

Proposition 1.4 For a space (X, τ) the following are equivalent:

(1) (X, τ) is strongly irresolvable,

(2) $PO(X, \tau) \subseteq SO(X, \tau)$,

(3) $SPO(X, \tau) = SO(X, \tau)$

(4) $BO(X, \tau) = SO(X, \tau)$.

Proof. (1) $\Leftrightarrow$ (2) has been shown in [5] and (1) $\Leftrightarrow$ (3) has been shown in [6]. Clearly, (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) follow immediately from Proposition 1.1.

Proposition 1.5 [6] For a space (X, τ) the following are equivalent:

(1) $SPO(X, \tau) = SO(X, \tau) \cup PO(X, \tau)$,

(2) (X, τ) is extremally disconnected or strongly irresolvable.

2 The results

We start by characterizing those spaces where the class of b-open sets coincides with the class of semi-preopen sets. The Hewitt-representation of a space (X, τ) will be denoted by $X = F \cup G$.

Theorem 2.1 For a space (X, τ) the following are equivalent:
(1) $BO(X, \tau) = SPO(X, \tau)$,
(2) $\overline{W}$ is open for every open resolvable subspace W,
(3) $\overline{G}$ is open and $\text{int}F \subseteq ED(X, \tau)$,
(4) (X, τ) is the topological sum of an extremally disconnected space and a strongly irresolvable space.

Proof. (1) $\Rightarrow$ (2): If $W = \emptyset$, we are done, so let W be a nonempty open resolvable subspace and let E_1 and E_2 be disjoint dense subsets of W. Pick $x \in \overline{W}$. If $x \in W$ then clearly $x \in \text{int}\overline{W}$. If $x \notin W$, let $S = E_1 \cup \{x\}$. It is easily checked that $\overline{S} = \overline{W}$, hence $\overline{\text{int}\overline{S}} = \overline{W}$ and so $S \in SPO(X, \tau)$. If U is an open set such that $U \subseteq S$, then $\emptyset = U \cap E_2 = U \cap \overline{E_2} = U \cap \overline{W} = U$. Thus $\text{int}S = \emptyset$. By hypothesis, $S \in BO(X, \tau)$ and so $x \in S \subseteq \overline{\text{int}S} \cup \text{int}\overline{S} = \text{int}\overline{S} = \text{int}\overline{W}$. Thus $\overline{W}$ is open.
(2) $\Rightarrow$ (3): Since $\text{int}F$ is an open resolvable subspace, $\overline{\text{int}F}$ is open by hypothesis. Since F is closed and $\overline{\text{int}F} \subseteq F$, we have $\text{int}F = \overline{\text{int}F}$ and so $\overline{G} = \text{int}\overline{G}$, i.e. $\overline{G}$ is open. Now let $x \in \text{int}F$ and let U be open such that $x \in \overline{U}$. If $W = U \cap \text{int}F$ then W is open and resolvable and $x \in \overline{W}$. By hypothesis, $x \in \text{int}\overline{W} \subseteq \text{int}\overline{U}$ and so x is an $e.d.$-point of (X, τ).
(3) $\Rightarrow$ (4): Clearly, $X = \text{intF} \cup \overline{G}$ and $\overline{G}$ is a clopen strongly irresolvable subspace and $\text{int}F$ is clopen and extremally disconnected as a subspace.
(4) $\Rightarrow$ (1): Let $X = E \cup C$ be the topological sum of an extremally disconnected subspace E and a strongly irresolvable subspace C. Let $S \in SPO(X, \tau)$. Since E and C are clopen, it is easily verified that $S \cap E \in SPO(E, \tau|_E)$, hence $S \cap E \in PO(E, \tau|_E)$ by Proposition 1.3 and so $S \cap E \in PO(X, \tau)$. In a similar manner, $S \cap C \in SPO(C, \tau|_C)$, hence $S \cap C \in SO(C, \tau|_C)$ by Proposition 1.4 and so $S \cap C \in SO(X, \tau)$. Since $S = (S \cap E) \cup (S \cap C)$, S is the union of a preopen set and a semi-open set and thus b-open, i.e. $S \in BO(X, \tau)$. $\square$

Corollary 2.2 Let (X,τ) be resolvable. Then $BO(X,\tau)=SPO(X,\tau)$ if and only if (X,τ) is extremally disconnected.

In our next result we characterize those spaces where the class of b-open sets coincides with $PO(X,\tau)\cup SO(X,\tau)$.

Theorem 2.3 For a space (X,τ) the following are equivalent:

(1) $PO(X,\tau)\cup SO(X,\tau)=BO(X,\tau)$,

(2) for each subset $A\subseteq X$, $\mathrm{int}\overline{A}\subseteq\overline{\mathrm{int}A}$ or $\overline{\mathrm{int}A}\subseteq\mathrm{int}\overline{A}$.

Proof. (1) $\Rightarrow$ (2): Let $A\subseteq X$ and let $B=(A\cap\mathrm{int}\overline{A})\cup\overline{\mathrm{int}A}$. It is easily checked that $\mathrm{int}\overline{A}=\mathrm{int}\overline{B}$ and that $\overline{\mathrm{int}A}=\overline{\mathrm{int}B}$ (see e.g. [6], Proposition 11). Since $A\cap\mathrm{int}\overline{A}\in PO(X,\tau)$ and $\overline{\mathrm{int}A}\in SO(X,\tau)$, we have that $B\in BO(X,\tau)$. By hypothesis, $B\in PO(X,\tau)\cup SO(X,\tau)$. If $B\in PO(X,\tau)$, then $\overline{\mathrm{int}A}\subseteq B\subseteq\mathrm{int}\overline{B}=\mathrm{int}\overline{A}$. If $B\in SO(X,\tau)$, then $A\cap\mathrm{int}\overline{A}\subseteq B\subseteq\overline{\mathrm{int}B}=\overline{\mathrm{int}A}$ and consequently $\mathrm{int}\overline{A}\subseteq\overline{\mathrm{int}\overline{A}}=\overline{A\cap\mathrm{int}\overline{A}}\subseteq\overline{\mathrm{int}A}$.

(2) $\Rightarrow$ (1): Let $S\in BO(X,\tau)$. If $\mathrm{int}\overline{S}\subseteq\overline{\mathrm{int}S}$, then $S\subseteq\overline{\mathrm{int}S}$ and so $S\in SO(X,\tau)$. If $\overline{\mathrm{int}S}\subseteq\mathrm{int}\overline{S}$, then $S\subseteq\mathrm{int}\overline{S}$ and so $S\in PO(X,\tau)$. $\square$

As a consequence we have the following new characterization of spaces (X,τ) where the class of semi-preopen sets coincides with $PO(X,\tau)\cup SO(X,\tau)$ (concerning other characterizations we refer the reader to [6]).

Corollary 2.4 For a space (X,τ) the following are equivalent:

(1) $PO(X,\tau)\cup SO(X,\tau)=SPO(X,\tau)$,

(2) $\overline{W}$ is open for every open resolvable subspace W, and for each subset $A\subseteq X$, $\mathrm{int}\overline{A}\subseteq\overline{\mathrm{int}A}$ or $\overline{\mathrm{int}A}\subseteq\mathrm{int}\overline{A}$.

It is obvious that the condition $PO(X,\tau)\cup SO(X,\tau)=SPO(X,\tau)$ implies that $PO(X,\tau)\cup SO(X,\tau)=BO(X,\tau)$. In our next example we point out that this implication cannot be reversed in general.

Example 2.5 Let X_1 and X_2 be disjoint infinite sets, let $p \notin X_1 \cup X_2$ and let $X = X_1 \cup X_2 \cup \{p\}$. A topology τ on X is defined in the following way: a basic neighbourhood of $x \in X_i$ is a cofinite subset of X_i , $i \in \{1,2\}$, a basic neighbourhood of p has the form $\{p\} \cup C_1 \cup C_2$ where C_i is a cofinite subset of X_i , $i \in \{1,2\}$.

Clearly (X,τ) is a T_1 space and X_1 and X_2 are open subspaces of (X,τ). If D_i and E_i are disjoint infinite subsets of X_i , $i \in \{1,2\}$, then $D_1 \cup D_2$ and $E_1 \cup E_2$ are disjoint dense subsets of (X,τ), and so (X,τ) is resolvable. Since $p \in \overline{X_1} \setminus \mathrm{int}\overline{X_1}$, (X,τ) is not extremally disconnected and so $SPO(X,\tau) \neq PO(X,\tau) \cup SO(X,\tau)$ by Proposition 1.5.

Now let $A \subseteq X$. If A is finite, then A is closed and so $\mathrm{int}\overline{A} = \mathrm{int}A \subseteq \overline{\mathrm{int}A}$. Hence, we assume that A is infinite. If both $A \cap X_1$ and $A \cap X_2$ are infinite, then A is dense and so $\overline{\mathrm{int}A} \subseteq \mathrm{int}\overline{A}$. So we may assume without loss of generality the case that $A \cap X_1$ is infinite and $A \cap X_2$ is finite. Then $p \notin \mathrm{int}A$, $\overline{A \cap X_1} = X_1 \cup \{p\}$ and $\overline{A \cap X_2} = A \cap X_2$. If $A \cap X_1$ is not cofinite in X_1 , then $\mathrm{int}A = \emptyset$ and so $\overline{\mathrm{int}A} \subseteq \mathrm{int}\overline{A}$. If $A \cap X_1$ is cofinite in X_1 , then $\mathrm{int}A = A \cap X_1$ and $\overline{\mathrm{int}A} = X_1 \cup \{p\}$. Furthermore, $\overline{A} = (X_1 \cup \{p\}) \cup (A \cap X_2)$ and $\mathrm{int}\overline{A} = X_1$, thus $\mathrm{int}\overline{A} \subseteq \overline{\mathrm{int}A}$.

Hence, for every subset $A \subseteq X$ we have $\mathrm{int}\overline{A} \subseteq \overline{\mathrm{int}A}$ or $\overline{\mathrm{int}A} \subseteq \mathrm{int}\overline{A}$, i.e. $BO(X,\tau) = PO(X,\tau) \cup SO(X,\tau)$ by Theorem 2.3. $\square$

We conclude our paper with a result that for a large class of spaces, however, the conditions "$PO(X,\tau) \cup SO(X,\tau) = SPO(X,\tau)$" and "$PO(X,\tau) \cup SO(X,\tau) = BO(X,\tau)$" coincide. Recall that a space (X,τ) is called quasi-regular [11] if every nonempty open set contains a nonempty regular closed set (i.e. the closure of a nonempty open set).

Theorem 2.6 For a quasi-regular space (X,τ) the following are equivalent:

(1) $PO(X,\tau) \cup SO(X,\tau) = SPO(X,\tau)$,

(2) $PO(X,\tau) \cup SO(X,\tau) = BO(X,\tau)$.

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (1): If (X,τ) is strongly irresolvable, then $SPO(X,\tau) = PO(X,\tau) \cup SO(X,\tau)$ by Proposition 1.5. So we assume that (X,τ) has a nonempty open resolvable subspace W. Let

E_1 and E_2 be disjoint dense subsets of W. We claim that (X,τ) is extremally disconnected. Suppose that (X,τ) is not extremally disconnected. Then there is a point $x \in X$ and an open set U such that $x \in \overline{U} \setminus \mathrm{int}\overline{U} = \overline{U} \cap \overline{X \setminus \overline{U}}$. Then $W \cap (U \cup (X \setminus \overline{U}) \neq \emptyset$, and without loss of generality we assume that $W \cap (X \setminus \overline{U}) = W_1 \neq \emptyset$. Since (X,τ) is quasi-regular, there is a nonempty open set V such that $\overline{V} \subseteq W_1$. Let $S = U \cup \{x\}$. Then S is semi-open but not preopen. Let $T = V \cap E_1$. Then T is preopen but not semi-open, since $\mathrm{int}T = \emptyset$. We obviously have $S \subseteq \overline{U}$ and $T \subseteq X \setminus \overline{U}$, and $T \subseteq \overline{V}$ and $S \subseteq X \setminus \overline{V}$.
If we set $A = S \cup T$, then clearly A is b-open. Hence, by assumption, A is preopen or semi-open. If A is preopen, then $A \cap (X \setminus \overline{V}) = S$ is preopen, a contradiction. If A is semi-open, then $A \cap (X \setminus \overline{U}) = T$ is semi-open, a contradiction. Thus (X,τ) is extremally disconnected and so $SPO(X,\tau) = PO(X,\tau) \cup SO(X,\tau)$ by Proposition 1.5. □

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