The θ-topology - some basic questions *

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Abstract

The purpose of this paper is to raise some very fundamental questions about the θ-topology which we encountered during our research. We acknowledge the possibility that some of our questions may have already been answered somewhere in the literature. Nevertheless we consider it to be quite useful to pose these basic questions (maybe again) as they appear in one form or another fairly often. We strongly hope that current researchers will come up with new and interesting answers.

1 Introduction and Preliminaries

Let (X, τ) be a topological space. We shall denote the closure and the interior of a subset $A \subseteq X$ by $\mathrm{cl}^\tau(A)$ and $\mathrm{int}^\tau(A)$, respectively, or simply by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ if there is no danger of confusion. $CO(X, \tau)$ denotes the family of all subsets of (X, τ) that are both open and closed in (X, τ). If $A \subseteq X$ then $|A|$ denotes the cardinality of A, and $\tau|_A$ will denote the subspace topology on A. For each $x \in X$, the families of all neighbourhoods (resp. all open neigbourhoods, all closed neighbourhoods) of x will be denoted by $\mathcal{U}(x)$ (resp. $\mathcal{U}^o(x)$, $\mathcal{U}^c(x)$). A space (X, τ) will be called regular if for each closed set $A \subseteq X$ and each point $x \notin A$ there exist open sets U and V such that $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.

The definition of the θ-topology has its origin in the famous paper of Veličko [8] on H-closed topological spaces. Recall that a space (X, τ) is called H-closed if every open cover of (X, τ) has a finite subfamily whose closures cover X . Let (X, τ) be a topological space and let

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$A \subseteq X$. A point $x \in X$ is called a θ-contact point [8] (resp. a δ-contact point [8]) of A if $A \cap \text{cl(U)} \neq \emptyset$ (resp. $A \cap \text{int(cl(A))} \neq \emptyset$) for every open set U containing x. The set of all θ-contact points (resp. δ-contact points) of $A \subseteq X$ is called the θ-closure [8] (resp. δ-closure [8]) of A and denoted by $\text{cl}_\theta(A)$ (resp. $\text{cl}_\delta(A)$). The set $A \subseteq X$ is said to be θ-closed [8] (resp. δ-closed [8]) if $\text{cl}_\theta(A) = A$ (resp. $\text{cl}_\delta(A) = A$). Complements of θ-closed sets (resp. δ-closed sets) are called θ-open (resp. δ-open). It has been pointed out in [8] that $\text{cl}(A) \subseteq \text{cl}_\theta(\text{A})$ and that $\text{cl}_\theta(\text{A}) \subseteq \text{cl}_\theta(\text{B})$ for any subsets $A \subseteq B \subseteq X$. Moreover, the families of all θ-open sets (resp. δ-open sets) are topologies on X, called the θ-topology (resp. δ-topology) and denoted by τ_θ (resp. τ_δ). It follows straightforward from the definitions that we have $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ and that $\tau_\theta = \tau$ if and only if (X, τ) is regular. Observe also that a subset $U \subseteq X$ is θ-open (resp. δ-open) if for each $x \in U$ there exists $V \in \tau$ such that $x \in V \subseteq \text{cl(V)} \subseteq \text{U}$ (resp. $x \in V \subseteq \text{int(cl(V))} \subseteq \text{U}$) . The space (X, τ_δ), having the regular open subsets of (X, τ) as a base, is frequently also called the semi-regularization of (X, τ).

Definition 1 A topological space (X, τ) is called

(i) semi-regular, if $\tau_\delta = \tau$,

(ii) almost regular [6], if for each regular closed set $F \subseteq X$ and each point $x \notin F$ there exist disjoint open sets containing F and x respectively,

(iii) locally indiscrete, if every open subset of (X, τ) is closed.

2 Very basic questions

The most immediate question that comes to mind is the question concerning the existence of non-trivial θ-open sets in a topological space, i.e. we want to know when the θ-topology fails to be indiscrete. While we have not been able to resolve this question in a fully satisfactory way, we do have some interesting results.

Definition 2 A topological space (X, τ) is said to have the (P)-property if for each collection $(O_i)_{i \in I}$ of open subsets satisfying $\bigcup_{i \in I} O_i \notin \{\emptyset, X\}$ we have that $\bigcup_{i \in I} O_i \neq \bigcup_{i \in I} \text{cl}(O_i)$.

Proposition 2.1 Let (X, τ) be a topological space. Then (X, τ_θ) is indiscrete if and only if (X, τ) has the (P)-property.

Proof. Suppose that (X, τ_θ) is indiscrete and that (X, τ) does not have the (P)-property. Then there is a collection $(O_i)_{i \in I}$ of τ-open sets with $\bigcup_{i \in I} O_i \notin \{\emptyset, X\}$ and $\bigcup_{i \in I} O_i = \bigcup_{i \in I} \mathrm{cl}(O_i)$. Let $V = \bigcup_{i \in I} O_i$. If $x \in V$ then there is some $i \in I$ such that $x \in O_i$. By assumption we clearly have $x \in O_i \subseteq \mathrm{cl}(O_i) \subseteq V$ and so $V \in \tau_\theta$, a contradiction.
To show the converse, suppose that (X, τ_θ) is not indiscrete. Then there exists $V \in \tau_\theta$ with $V \notin \{\emptyset, X\}$. For each $x \in V$ there exists some τ-open set O_x such that $\mathrm{cl}(O_x) \subseteq V$. Clearly we have that $V = \bigcup_{x \in V} O_x = \bigcup_{x \in V} \mathrm{cl}(O_x)$ and so (X, τ) does not have the (P)-property.

Proposition 2.2 Let (X, τ) be a topological space. Suppose there is some $x \in X$ such that $\mathcal{U}^c(x) \setminus \{X\}$ is a base for $\mathcal{U}(x)$. Then (X, τ_θ) is not indiscrete.

Proof. We write the (nonempty) family $\mathcal{U}^c(x) \setminus \{X\}$ in the form $(W_i)_{i \in I}$ and claim that $\bigcap_{i \in I} W_i = \bigcap_{i \in I} \mathrm{int}(W_i)$. Suppose that $y \in \bigcap_{i \in I} W_i$ and there exists $j \in I$ such that $y \notin \mathrm{int}(W_j)$. Since $\mathrm{int}(W_j)$ is an open neighbourhood of x, there exists $k \in I$ such that $W_k \subseteq \mathrm{int}(W_j)$ and so $y \notin W_k$, a contradiction. Thus our claim is proved. Now let $O_i = X \setminus W_i$ for each $i \in I$. Then $(O_i)_{i \in I}$ is a collection of τ-open sets satisfying $\bigcup_{i \in I} O_i \notin \{\emptyset, X\}$ and $\bigcup_{i \in I} O_i = \bigcup_{i \in I} \mathrm{cl}(O_i)$. Hence (X, τ) does not have the (P)-property. By Proposition 2.1, (X, τ_θ) fails to be indisrete.

Proposition 2.3 Let (X, τ) be a topological space with $|X| > 1$. Then (X, τ_θ) is indiscrete if and only if for each subset $T \subseteq X$ with $T \neq X$ there exists some $x \notin T$ such that T and $\{x\}$ cannot be separated by open sets.

Proof. Suppose that (X, τ_θ) is indiscrete. Let $T \subseteq X$ with $T \notin \{\emptyset, X\}$. Hence, if $O = X \setminus T$ then $O \notin \tau_\theta$ and so there exists $x \in O$ such that for no τ-open set V we have that $x \in V \subseteq \mathrm{cl}(V) \subseteq O$. But this means that T and $\{x\}$ cannot be separated by open sets.
To show the converse, suppose that (X, τ_θ) is not indiscrete. Then there exists a subset $O \in \tau_\theta$ with $O \notin \{\emptyset, X\}$. Let $T = X \setminus O$. If $x \in O$, i.e. if $x \notin T$, there is a τ-open set V

such that $x \in V \subseteq \mathrm{cl}(V) \subseteq O$. Clearly $x \in V$ and $T \subseteq X \setminus \mathrm{cl}(V)$, and so T and $\{x\}$ can be separated by open sets.

Recall that a space (X, τ) is said to be hyperconnected if every open subset is dense. It is very well known that (X, τ) is hyperconnected if and only if (X, τ_δ) is indiscrete. We shall say that a point $p \in X$ is a point of hyperconnectedness if every (open) neighbourhood of p is dense.

Proposition 2.4 Let (X, τ) be a topological space such that $p \in X$ is a point of hyperconnectedness. Then (X, τ_θ) is indiscrete.

Proof. Let $W \in \tau_\theta$ be nonempty and let $x \in W$. Hence there is a τ-open set U_x such that $x \in U_x \subseteq \mathrm{cl}(U_x) \subseteq W$. Since every neighbourhood of p is dense it follows that $p \in \mathrm{cl}(U_x)$ and so $p \in W$. So there has to be a τ-open set V_p such that $p \in V_p \subseteq \mathrm{cl}(V_p) \subseteq W$. Consequently, $W = X$.

Remark 2.5 Consider the space (X, τ) from Example 2.5 in [1]. Proposition 2.4 immediately shows that (X, τ_θ) has to be indiscrete while it is easily established that (X, τ_δ) fails to be indiscrete. Hence the fact that (X, τ_θ) is indiscrete does not imply that (X, τ_δ) must be indiscrete.

Question 1 What are some other necessary and sufficient conditions on a space (X, τ) such that (X, τ_θ) has to be indiscrete?

We shall next consider the relationships between τ, τ_δ and τ_θ. It is clear that $CO(X, \tau_\theta) = CO(X, \tau) \subseteq \tau_\theta \subseteq \tau_\delta \subseteq \tau$. Furthermore, $\tau_\delta = \tau$ if and only if (X, τ) is semi-regular, $\tau_\theta = \tau_\delta$ if and only if (X, τ) is almost regular (see [2] and [4]), $\tau_\theta = \tau$ if and only if (X, τ) is regular, and $CO(X, \tau) = \tau$ if and only if (X, τ) is locally indiscrete. We also observe that $CO(X, \tau) = \tau_\theta$ if and only if (X, τ_θ) is locally indiscrete. This leads to the question which property the space (X, τ) has to satisfy to make (X, τ_θ) locally indiscrete.

Definition 3 A topological space (X,τ) is said to have the $(P1)$-property if for each collection $(O_i)_{i\in I}$ of open subsets satisfying $\bigcup_{i\in I} O_i = \bigcup_{i\in I} \mathrm{cl}(O_i)$ we have that $\bigcup_{i\in I} O_i$ is closed.

Proposition 2.6 For a space (X,τ) the following are equivalent:
(1) $CO(X,\tau) = \tau_\theta$,
(2) (X,τ) has the $(P1)$-property.

Proof. (1) $\Rightarrow$ (2) : Suppose that (X,τ) does not have the $(P1)$-property. Then there is a collection $(O_i)_{i\in I}$ of τ-open subsets with $\bigcup_{i\in I} O_i = \bigcup_{i\in I} \mathrm{cl}(O_i)$ such that $\bigcup_{i\in I} O_i$ is not closed. If $V = \bigcup_{i\in I} O_i$ then clearly $V \in \tau_\theta$, and so, by assumption, V has to be τ-closed, a contradiction.
(2) $\Rightarrow$ (1) : We have to show that $\tau_\theta \subseteq CO(X,\tau)$. Let $V \in \tau_\theta$ and for each $x \in V$ let U_x be a τ-open set with $x \in U_x \subseteq \mathrm{cl}(U_x) \subseteq V$. We clearly have that $V = \bigcup_{x\in V} U_x = \bigcup_{x\in V} \mathrm{cl}(U_x)$ and so, by assumption, V has to be also τ-closed.

Now let (X,τ) be a topological space, $W \in \tau_\theta$ and $y \in cl^\tau(W)$. Suppose that V is a finite τ-open neighbourhood of y and that $y \notin W$. Then $V \cap W$ is nonempty and finite. For each $x \in V \cap W$ there is a τ-open set U_x such that $x \in U_x \subseteq \mathrm{cl}(U_x) \subseteq W$, and, since $y \notin \mathrm{cl}(U_x)$, there is a τ-open set V_x such that $y \in V_x$ and $V_x \cap U_x = \emptyset$. Clearly, $V \cap \bigcap_{x\in V\cap W} V_x$ is a τ-open neighbourhood of y having empty intersection with W, a contradiction. Hence $y \in W$. As a consequence we have

Proposition 2.7 Let (X,τ) be a topological space and suppose that each $x \in X$ has a finite neighbourhood. Then $CO(X,\tau) = \tau_\theta$.

It is very well known that for every topological space (X,τ) we have $(\tau_\delta)_\delta = \tau_\delta$. What can be said about the corresponding relation $(\tau_\theta)_\theta = \tau_\theta$? Clearly, $(\tau_\theta)_\theta = \tau_\theta$ if and only if (X,τ_θ) is regular. Recall that a space (X,τ) is said to be Urysohn if for each pair $x, y \in X$ with $x \neq y$ there exist open sets U and V such that $x \in U$, $y \in V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$.

Proposition 2.8 For a topological space (X,τ), in general $(\tau_\theta)_\theta \neq \tau_\theta$ holds.

Proof. Let (X, τ) be a space that is Hausdorff but not Urysohn (see e.g. [7]). Observe that (X, τ_θ) is T_1, i.e. singletons are τ_θ-closed, since (X, τ) is Hausdorff. Suppose that (X, τ_θ) is regular and let $x, y \in X$ with $x \neq y$. Hence there exist $V_x, V_y \in \tau_\theta$ such that $x \in V_x$, $y \in V_y$ and $V_x \cap V_y = \emptyset$. Consequently there exist τ-open sets O_x and O_y with $x \in O_x \subseteq \mathrm{cl}(O_x) \subseteq V_x$ and $y \in O_y \subseteq \mathrm{cl}(O_y) \subseteq V_y$ and so $\mathrm{cl}(O_x) \cap \mathrm{cl}(O_y) = \emptyset$. But this shows that (X, τ) is Urysohn, a contradiction. Observe also, that (X, τ_θ) fails to be regular.

Question 2 Characterize in additional ways the spaces (X, τ) such that $(\tau_\theta)_\theta = \tau_\theta$.

Remark 2.9 The referee pointed out that $(\tau_\theta)_\theta = \tau_\theta$ whenever (X, τ) is almost regular. To see this, let (X, τ) be almost regular. Then $\tau_\theta = \tau_\delta$ (see [2] and [4]) and (X, τ_δ) is regular. Consequently, $(\tau_\theta)_\theta = (\tau_\delta)_\theta = \tau_\delta = \tau_\theta$.

3 On identifying and constructing θ-open sets

In general it seems to be fairly difficult to describe the family of θ-open sets of a given space (X, τ). Moreover, effective ways to construct θ-open subsets in a (non-regular) space seem to be rare. By utilizing the concept of a simple extension of a space (X, τ) (due to N. Levine [3]) there is at least one non-trivial way to work with the θ-topology. Let (X, τ) be a topological space and let $D \subseteq X$ be a non-open subset. The simple extension of τ by D is the (strictly) finer topology σ defined by $\sigma = \{O \cup (U \cap D) \ : \ O, U \in \tau\}$. Levine [3] pointed out that $\mathrm{cl}^\sigma(A) = \mathrm{cl}^\tau(A) \cap ((X \setminus D) \cup (D \cap \mathrm{cl}^\tau(A \cap D)))$ and $\mathrm{cl}^\sigma(A \cap D) = \mathrm{cl}^\tau(A \cap D)$ for any subset $A \subseteq X$. Moreover, if D is dense in (X, τ) then (X, σ) fails to be regular. The following observation is probably well known and easily proved.

Lemma 3.1 Let τ and σ be topologies on X with $\tau \subseteq \sigma$. Then $\tau_\theta \subseteq \sigma_\theta$.

Proposition 3.2 Let (X, τ) be regular and let $D \subseteq X$ be non-open and dense. If σ denotes the simple extension of τ by D then $\tau = \sigma_\theta$.

Proof. Since (X,τ) is regular we have $\tau_\theta = \tau \subseteq \sigma$ and by Lemma 3.1 we conclude that $\tau \subseteq \sigma_\theta$.

Now let $U \in \sigma_\theta$ and let $x \in U$. Then there exists $V \in \sigma$ such that $x \in V \subseteq \mathrm{cl}^\sigma(V) \subseteq U$. Let $O, O^* \in \tau$ such that $V = O \cup (O^* \cap D)$. Since D is dense in (X,τ), it is easily checked that $\mathrm{cl}^\tau(V \cap D) = \mathrm{cl}^\tau(V) = \mathrm{cl}^\tau(O \cup O^*)$ and thus $\mathrm{cl}^\tau(V) = \mathrm{cl}^\sigma(V)$. Hence, if $W = O \cup O^*$, then $W \in \tau$, $x \in W$ and $\mathrm{cl}^\tau(W) \subseteq U$, proving that $U \in \tau$.

Example 3.3 Let $X = [0,1] \subseteq \mathbb{R}$ be the unit interval equipped with the usual topology denoted by τ. Let D be a non-open dense subset of (X,τ) and let σ be the simple extension of τ by D. Then (X,σ) is a non-regular Hausdorff space which, of course, cannot be compact. However, (X,σ_θ) obviously is a compact Hausdorff space (and thus normal), since $\sigma_\theta = \tau$.

4 Some topological properties

There are, of course, numerous topological properties that are worth to be discussed with respect to the θ-topology. We have picked out a few of them along with some interesting questions.

Proposition 4.1 Let (X,τ) be a topological space. Then (X,τ_θ) is connected if and only if (X,τ) is connected.

Proof. Follows from the fact that $CO(X,\tau_\theta) = CO(X,\tau)$.

Concerning separation axioms we first observe the following result whose proof is elementary.

Proposition 4.2 Let (X,τ) be a topological space. Then (X,τ) is Hausdorff if and only if (X,τ_θ) is T_1 if and only if (X,τ_θ) is T_0.

Example 4.3 Consider the space (X,τ) of Example 75 in [7], i.e. the irrational slope topology. Then (X,τ) and thus (X,τ_δ) are Hausdorff while (X,τ_θ) is T_1 by Proposition 4.2. Since any two nonempty regular closed sets have nonempty intersection, (X,τ_θ) cannot be Hausdorff.

Clearly, if (X, τ_θ) is Hausdorff, then (X, τ) has to be Urysohn. Therefore, the following two natural questions arise.

Question 3 Is there an Urysohn space (X, τ) such that (X, τ_θ) fails to be Hausdorff?

Question 4 Find necessary and sufficient conditions on a space (X, τ) such that (X, τ_θ) is Hausdorff.

Concerning subspaces we offer the following result.

Proposition 4.4 Let (X, τ) be a topological space and let $A \subseteq X$. Then $(\tau_\theta)|_A \subseteq (\tau|_A)_\theta$.

Proof. Let $V \in (\tau_\theta)|_A$. Then there exists $U \in \tau_\theta$ such that $V = U \cap A$. For each $x \in V$ there exists $U_x \in \tau$ such that $x \in U_x \subseteq \mathrm{cl}^\tau(U_x) \subseteq U$. If $V_x = U_x \cap A$, then $x \in V_x$ and $V_x \in \tau|_A$. Thus $x \in V_x \subseteq \mathrm{cl}^{\tau|_A}(V_x) = \mathrm{cl}^\tau(V_x) \cap A = \mathrm{cl}^\tau(U_x \cap A) \cap A \subseteq V$. Consequently $V \in (\tau|_A)_\theta$.

The following example shows that the inclusion in Proposition 4.4 cannot be reserved.

Example 4.5 Let $X = \mathbb{R} \cup Y$ where $\mathbb{R}$ denotes the set of reals and Y is an infinite set disjoint from $\mathbb{R}$. We define a topology τ on X in the following way: any cofinite subset of Y is open and a basic open neighbourhood of a point $x \in \mathbb{R}$ is the union of an open interval of $\mathbb{R}$ and a cofinite subset of Y. Since each point of Y is a point of hyperconnectedness, (X, τ_θ) has to be indiscrete by Propostion 2.4. Now let $A = \mathbb{R}$. Then $(\tau_\theta)|_A$ is clearly indiscrete. Obviously, $\tau|_A$ is the usual topology on $A = \mathbb{R}$ and so $(\tau|_A)_\theta = \tau|_A$, showing that $(\tau_\theta)|_A \neq (\tau|_A)_\theta$.

Question 5 Let (X, τ) be a topological space. Characterize the subspaces $A \subseteq X$ such that $(\tau_\theta)|_A = (\tau|_A)_\theta$.

Question 6 Characterize the spaces (X, τ) such that for every subspace $A \subseteq X$ we have $(\tau_\theta)|_A = (\tau|_A)_\theta$.

Question 7 Let (X, τ) be a topological space and let $\tau \times \tau$ be the product topology on $X \times X$. What is the relationship between the topologies $(\tau \times \tau)_\theta$ and $\tau_\theta \times \tau_\theta$?

5 On the θ-closure operator

In [8] Veličko already noted that the θ-closure $\mathrm{cl}_\theta(A)$ of a subset A of a space (X,τ) does not define a Kuratowski closure operator in general, hence cannot be the closure of A with respect to (X,τ_θ) which we shall denote by $\overline{A}^\theta$. It is easily checked that $\mathrm{cl}_\theta(A) \subseteq \overline{A}^\theta$ but this implication cannot be reversed in general (see [5], page 313). In fact, cl_θ is a Kuratowski closure operator if and only if it is idempotent, i.e. if $\mathrm{cl}_\theta(\mathrm{cl}_\theta(A)) = \mathrm{cl}_\theta(A)$ for every subset $A \subseteq X$. Of course, this leads to the question which conditions a space must satisfy so that cl_θ is a Kuratowski closure operator.

Proposition 5.1 (see [2]) A space (X,τ) is almost regular if and only if cl_θ is idempotent.

If we restrict ourselves to H-closed topological spaces, the following results are available in the literature.

Proposition 5.2 Let (X,τ) be Hausdorff and H-closed.
(1) [8] If (X,τ) is in addition Urysohn, then cl_θ is a Kuratowski closure operator.
(2) [5] If (X,τ) fails to be Urysohn, then cl_θ need not be a Kuratowski closure operator.

Suppose that (X,τ) is regular but not Hausdorff. Then cl_θ clearly is a Kuratowski closure operator as it coincides with the usual closure. Hence such a space need not necessarily be Urysohn. However, it has been shown in [6], Theorem 3.2, that every almost regular Hausdorff space is Urysohn.

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