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ON SOME VERY STRONG COMPACTNESS CONDITIONS

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Abstract. The aim of this paper is to consider compactness notions by utilizing Λ-sets, V-sets, locally closed sets, locally open sets, λ-closed sets and λ-open sets. We completely characterize these variations of compactness, and also provide various interesting examples that support our results.

1. Introduction and preliminaries

In recent years certain types of subsets of a topological space have been considered and found to be useful. Maki [6] introduced the notion of a Λ-set and a V-set in a topological space and studied the associated closure operators. Ganster and Reilly [4] utilized locally closed subsets to prove a new decomposition of continuity. In [1], Arenas, Dontchev and Ganster considered λ-sets to obtain some other decompositions of continuity. Since then, numerous other authors have used these concepts in their work.

The purpose of our paper is to consider these and some related types of subsets of a topological space and investigate the associated compactness notions. To be more precise, we now define explicitly the notions that we shall work with.

Definition 1. For a subset A of a topological space (X,τ), the kernel of A is defined as $\ker A = \bigcap\{U : U \text{ is open and } A \subseteq U\}$.

Definition 2. A subset A of a topological space (X,τ) is called
(a) a Λ-set [6], if $A = \ker A$, i.e. if A is the intersection of open sets,
(b) a V-set [6], if $X \setminus A$ is a Λ-set, i.e. if A is the union of closed sets,
(c) locally closed [2], if $A = U \cap F$ where U is open and F is closed,

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(d) locally open, if $X \setminus A$ is locally closed, i.e. A is the union of an open and a closed set,

(e) λ-closed [1], if $A = U \cap F$ where U is a Λ-set and F is closed,

(f) λ-open, if $X \setminus A$ is λ-closed, i.e. the union of a V-set and an open set.

2. Strong notions of compactness

Definition 3. Let $\mathcal{A}$ be a family of subsets of (X,τ) such that $\bigcup\{A : A \in \mathcal{A}\} = X$. The space (X,τ) is called $\mathcal{A}$-compact if every cover of X by elements of $\mathcal{A}$ has a finite subcover.

Of course, if $\mathcal{A} = \tau$, then $\mathcal{A}$-compactness coincides with the usual compactness. If $\mathcal{A}$ denotes the family of closed subsets of a space (X,τ), then (X,τ) is $\mathcal{A}$-compact if and only if (X,τ) is strongly S-closed.

Definition 4. A space (X,τ) is called strongly S-closed [3] if every cover of X by closed sets has a finite subcover.

Definition 5. A space (X,τ) is called Λ-compact (resp. V-compact, LC-compact, LO-compact, λC-compact, λO-compact) if every cover of (X,τ) by Λ-sets (resp. V-sets, locally closed sets, locally open sets, λ-closed sets, λ-open sets) has a finite subcover.

It follows straight from the definition that each λC-compact space is both LC-compact and Λ-compact, and that each λO-compact space is both LO-compact and V-compact.

We shall first consider Λ-compact spaces. Clearly every Λ-compact space has to be compact. Observe that if (X,τ) is T_1, then each singleton is a Λ-set and therefore every Λ-compact T_1 space has to be finite.

Theorem 2.1. *For a space (X,τ) the following are equivalent*:

(1) *(X,τ) is Λ-compact*,

(2) *there exists a finite subset $\{x_1,\dots,x_k\} \subseteqq X$ such that $X = \ker\{x_1\} \cup \dots \cup \ker\{x_k\}$*.

Proof. (1) $\Rightarrow$ (2). Since $\{\ker\{x\} : x \in X\}$ is a cover of X by Λ-sets, there exists a finite subset $\{x_1,\dots,x_k\} \subseteqq X$ such that $X = \ker\{x_1\} \cup \dots \cup \ker\{x_k\}$.

(2) $\Rightarrow$ (1). Let $\mathcal{O}$ be a cover of X by Λ-sets. For each $i \in \{1,\dots,k\}$ there exists $O_i \in \mathcal{O}$ such that $x_i \in O_i$. Clearly $\ker\{x_i\} \subseteqq O_i$ and therefore $X = O_1 \cup \dots \cup O_k$, i.e. (X,τ) is Λ-compact. $\square$

Our next result shows that the class of V-compact spaces coincides with the class of strongly S-closed spaces.

Theorem 2.2. *For a space (X,τ) the following are equivalent*:

(1) *(X,τ) is V-compact*,

(2) (X,τ) *is strongly* S*-closed*,
(3) (X,τ) *has a finite dense subset.*

Proof. (1) $\Rightarrow$ (2) is obvious, and (2) $\Leftrightarrow$ (3) has been shown in [3].

(3) $\Rightarrow$ (1). Let $\{V_\alpha : \alpha \in I\}$ be a cover of (X,τ) by V-sets and let $D = \{x_1,\dots,x_k\}$ be a finite dense subset. For $i = 1,\dots,k$ pick $\alpha_i \in I$ such that $x_i \in V_{\alpha_i}$. Since V_{α_i} is a V-set, we have $\overline{\{x_i\}} \subseteqq V_{\alpha_i}$ for each $i = 1,\dots,k$. It follows that $X = V_{\alpha_1} \cup \dots \cup V_{\alpha_k}$ and we are done. $\square$

For the following we shall consider for a given space (X,τ) the family $\mathcal{S} = \{\, A \subseteqq X : A \text{ is open or closed in } (X,\tau)\}$. Clearly $\mathcal{S}$ is subbase for a topology τ^* on X, and the locally closed sets of (X,τ) form a base for τ^*. Moreover, it is easily checked that the topology generated by the locally open sets of (X,τ) coincides with τ^*. By Alexander's subbase theorem we now have

Theorem 2.3. *(X,τ^*) is compact if and only if (X,τ) is LC-compact if and only if (X,τ) is LO-compact.*

Lemma 2.4. *Let $\mathcal{A}$ be a family of subsets of a space (X,τ) such that $\mathcal{S} \subseteqq \mathcal{A}$. If (X,τ) is $\mathcal{A}$-compact, then (X,τ) is hereditarily compact and each closed subspace is strongly S-closed.*

Proof. Let $S \subseteqq X$ and let $\{O_\alpha : \alpha \in I\}$ be an open cover of S. If $O = \bigcup\{O_\alpha : \alpha \in I\}$ then O is open and $X = \bigcup\{O_\alpha : \alpha \in I\} \cup \{X \setminus O\}$. Since (X,τ) is $\mathcal{A}$-compact, there exist $\alpha_1,\dots,\alpha_k \in I$ such that $X = O_{\alpha_1} \cup \dots \cup O_{\alpha_k} \cup (X \setminus O)$ and so $S \subseteqq O_{\alpha_1} \cup \dots \cup O_{\alpha_k}$. Thus (X,τ) is hereditarily compact. Now let $F \subseteqq X$ be closed. Then $X = \bigcup\{\, \overline{\{x\}} : x \in F\} \cup \{X \setminus F\}$ and hence, since (X,τ) is $\mathcal{A}$-compact, there exist $x_1,\dots,x_k \in F$ such that $X = \overline{\{x_1\}} \cup \dots \cup \overline{\{x_k\}} \cup (X \setminus F)$. Clearly, $D = \{x_1,\dots,x_k\}$ is a finite dense subset of F and thus F is strongly S-closed. $\square$

The converse of Lemma 2.4 does not hold. By Example 3.4 there exists an infinite LC-compact space (X,τ). By Theorem 2.6, (X,τ) is hereditarily compact and each closed subspace is strongly S-closed. If we take $\mathcal{A} = \mathcal{P}(X)$ to be the power set of X then (X,τ) clearly fails to be $\mathcal{A}$-compact.

Remark 2.5. A comprehensive study of hereditarily compact spaces can be found in a paper of A. H. Stone [7]. Among many other results he showed that a space (X,τ) is hereditarily compact if and only if there exists no strictly decreasing sequence of closed sets.

Theorem 2.6. *For a space (X,τ) the following are equivalent*:
(1) (X,τ) *is* λO*-compact*,
(2) (X,τ) *is* LO*-compact*,
(3) (X,τ) *is* LC*-compact*,

(4) (X,τ) *is hereditarily compact and each closed subspace is strongly S-closed.*

PROOF. (1) $\Rightarrow$ (2) is obvious, since every locally open subset is also λ-open.

(2) $\Leftrightarrow$ (3) has been pointed out in Theorem 2.3, and (3) $\Rightarrow$ (4) follows from Lemma 2.4.

(4) $\Rightarrow$ (1). Let $\{A_i : i \in I\}$ be a cover of (X,τ) by λ-open sets, and for each $i \in I$ let $A_i = V_i \cup O_i$ where V_i is a V-set and O_i is open. If $O = \bigcup\{O_i : i \in I\}$ then O is open. Since (X,τ) is hereditarily compact, there exists a finite subset $I_1 \subseteqq I$ such that $O = \bigcup\{O_i : i \in I_1\}$. Furthermore, since each closed subspace of (X,τ) is strongly S-closed, there exist $x_1,\dots,x_n \in X$ such that $X \setminus O = \overline{\{x_1\}} \cup \dots \cup \overline{\{x_n\}}$. For each $j = 1,\dots,n$ pick A_{i_j} such that $x_j \in A_{i_j}$. Then $x_j \in V_{i_j}$ and thus $\overline{\{x_j\}} \subseteqq V_{i_j} \subseteqq A_{i_j}$. Consequently we have that $X = \bigcup\{A_i : i \in I_1\} \cup A_{i_1} \cup \dots \cup A_{i_n}$ proving that (X,τ) is λO-compact. □

We shall now consider λC-compactness which is evidently the strongest compactness notion that we discuss here. Given a space (X,τ) we first observe that if $A_x = \ker\{x\} \cap \overline{\{x\}}$ for some $x \in X$, then A_x is λ-closed. Now let $O \subseteqq X$ be open such that $O \cap A_x \neq \emptyset$. Since $O \cap \overline{\{x\}} \neq \emptyset$, we have $x \in O$ and thus $\ker\{x\} \subseteqq O$. Hence $O \cap A_x = O \cap \ker\{x\} \cap \overline{\{x\}} = A_x$ showing that A_x is an indiscrete subspace.

THEOREM 2.7. *For a space (X,τ) the following are equivalent:*
(1) (X,τ) *is λC-compact,*
(2) (X,τ) *is the finite union of indiscrete spaces,*
(3) *the topology τ is finite.*

PROOF. (1) $\Rightarrow$ (2). For each $x \in X$ let $A_x = \ker\{x\} \cap \overline{\{x\}}$. Then each A_x is λ-closed and $X = \bigcup\{A_x : x \in X\}$. By assumption, there exist $x_1, x_2, \dots, x_n \in X$ such that $X = A_{x_1} \cup \dots \cup A_{x_n}$. Thus (X,τ) is the finite union of indiscrete subspaces.

(2) $\Rightarrow$ (3). Let $X = A_1 \cup \dots \cup A_n$, where each A_i is indiscrete. If $O \subseteqq X$ is open, then $O = \bigcup\{O \cap A_i : i = 1,\dots,n\}$. Since each $O \cap A_i$ is either empty or A_i, O is the finite union of some A_i's. It follows that the topology has to be finite.

(3) $\Rightarrow$ (1). If τ is finite there are only finitely many Λ-sets and only finitely many closed sets, hence only finitely many λ-closed sets. Thus (X,τ) is λC-compact. □

We now provide an additional characterization of λC-compact spaces. First observe that a straightforward application of Zorn's lemma shows that each indiscrete subspace of a space (X,τ) is contained in a maximal indiscrete subspace. Next recall the following interesting result of Ginsburg and Sands [5].

PROPOSITION 2.8 [5]. *Every infinite topological space* (X, τ) *has an infinite subspace which is homeomorphic to* $\mathbb{N}$ *endowed with one of the following five topologies*: *the discrete topology*, *the indiscrete topology*, *the cofinite topology*, *the initial segment topology* (*see Example* 3.3) *and the final segment topology* (*see Example* 3.4). *In addition*, *the indiscrete topology on an infinite set is the only topology which is both compact and strongly* S*-closed.*

PROPOSITION 2.9. *Let* (X, τ) *be an infinite space where every infinite subspace contains an infinite indiscrete subspace. Then* (X, τ) *is the union of finitely many indiscrete subspaces and thus* λC*-compact.*

PROOF. We first show that there are only finitely many maximal indiscrete subspaces. Suppose that we have infinitely many distinct maximal indiscrete subspaces $\{B_n : n \in \mathbb{N}\}$. Then $B_n \cap B_m = \emptyset$ whenever $n \neq m$. For each $n \in \mathbb{N}$ pick $x_n \in B_n$. By assumption, $\{x_n : n \in \mathbb{N}\}$ has an indiscrete infinite subspace C. Hence there must be distinct $n, m \in \mathbb{N}$ such that $C \cap B_n \neq \emptyset$ and $C \cap B_m \neq \emptyset$. Since B_n and B_m are maximal indiscrete subspaces we have $C \subseteqq B_n$ and $C \subseteqq B_m$ and so $B_n \cap B_m \neq \emptyset$, a contradiction. Thus there are only finitely many maximal indiscrete subspaces.

Now let $A = \bigcup\{B : B$ is maximal indiscrete$\}$. We claim that $X \setminus A$ is finite. Suppose that $X \setminus A$ is infinite. By assumption, there is an infinite indiscrete subspace $C \subseteqq X \setminus A$. Pick a maximal indiscrete subspace C_1 such that $C \subseteqq C_1$. Then $C_1 \subseteqq A$ and so $C \cap A \neq \emptyset$, a contradiction. Thus $X \setminus A$ is finite and, of course, the finite union of indiscrete subspaces. Consequently, (X, τ) is the finite union of indiscrete subspaces. □

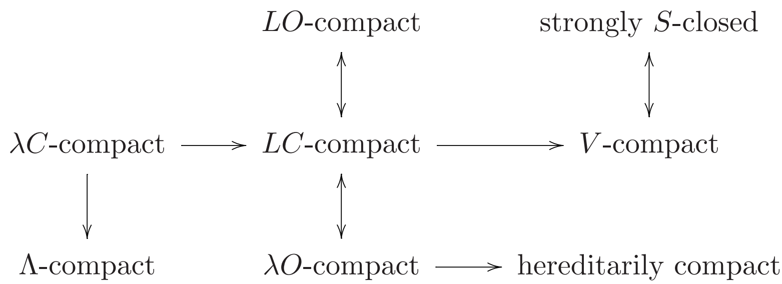
THEOREM 2.10. *For a space* (X, τ) *the following are equivalent*:
(1) (X, τ) *is* λC*-compact*,
(2) (X, τ) *is hereditarily compact and hereditarily strongly* S*-closed.*

PROOF. (1) $\Rightarrow$ (2). If (X, τ) is λC-compact, then τ is finite by Theorem 2.7 and thus (X, τ) clearly is hereditarily compact and hereditarily strongly S-closed.

(2) $\Rightarrow$ (1). If (X, τ) is finite we are done, so let (X, τ) be infinite. Let C be an infinite subspace of (X, τ). Then C is also hereditarily compact and hereditarily strongly S-closed. By Proposition 2.8, C must contain an infinite indiscrete subspace. By Proposition 2.9, (X, τ) is λC-compact.

3. Examples

The following diagram summarizes the relationships that either follow straightforward from the definitions or have been obtained in Section 2.



In the following examples we show that none of the implications can be reversed. Moreover, no additional implications hold. To avoid trivialities, X will always denote an infinite set.

Example 3.1 (the cofinite topology). Let τ be the cofinite topology on X. Then (X,τ) is hereditarily compact. Since (X,τ) is T_1, it is neither Λ-compact nor V-compact, and thus fails to be LC-compact.

Example 3.2 (the point-generated topology). Let $p \in X$ and let $\tau = \{\emptyset\} \cup \{O \subseteqq X : p \in O\}$. We have $\ker\{p\} = \{p\}$ and $\ker\{x\} = \{x,p\}$ whenever $x \neq p$, and so (X,τ) fails to be Λ-compact. Since $\overline{\{p\}} = X$ and $\overline{\{x\}} = \{x\}$ whenever $x \neq p$, (X,τ) is strongly S-closed and thus V-compact. The subspace $X \setminus \{p\}$ is closed and discrete, and thus (X,τ) cannot be hereditarily compact. Hence (X,τ) fails to be LC-compact by Theorem 2.6.

Example 3.3 (the initial segment topology on a limit ordinal). Let $X = \{\beta \in \mathrm{Ord} : \beta < \alpha\}$ where α denotes some (infinite) limit ordinal, and let $\tau = \{\emptyset\} \cup \{\,[0,\beta) : \beta < \alpha\}$. Since $\overline{\{\beta\}} = [\beta,\alpha)$ we have that (X,τ) is even hereditarily strongly S-closed and so, in particular, V-compact. Since (X,τ) fails to be compact, it is neither Λ-compact nor, by Theorem 2.6, LC-compact.

Example 3.4 (the final segment topology on an ordinal). Let $X = \{\beta \in \mathrm{Ord} : \beta < \alpha\}$ where α denotes some (infinite) ordinal, and let $\tau = \{X\} \cup \{\,[\beta,\alpha) : \beta < \alpha\}$. By Remark 2.5 we conclude that (X,τ) is hereditarily compact. Since $\ker\{0\} = X$, (X,τ) is also Λ-compact. Since $\overline{\{\beta\}} = [0,\beta]$ for $\beta < \alpha$, (X,τ) fails to be strongly S-closed ($= V$-compact) whenever α is a limit ordinal. In that case, (X,τ) cannot be LC-compact by Theorem 2.6.

In particular, let $\alpha = \omega + 1$, where ω denotes the set of finite ordinals. Since $\overline{\{\omega\}} = X$, we have that (X,τ) is strongly S-closed. Observe also that each proper closed subspace is finite and thus strongly S-closed. By Theorem 2.6, (X,τ) is LC-compact but fails to be λC-compact by Theorem 2.7.

Example 3.5 (another example of an LC-compact space). Let τ_1 be the cofinite topology on X and let τ_2 be the point-generated topology on X with respect to a point $p \in X$. Let $\tau = \tau_1 \cap \tau_2$. Then (X,τ) is clearly hereditarily compact and strongly S-closed. Since each proper closed subspace is finite and thus strongly S-closed, (X,τ) is LC-compact by Theorem 2.6. Observe that $\ker\{p\} = \{p\}$ and $\ker\{x\} = \{x,p\}$ whenever $x \neq p$. Thus (X,τ) fails to be Λ-compact and hence also cannot be λC-compact.

Example 3.6 (a Λ-compact, strongly S-closed space that is not LC-compact). Let $p,q \in X$ such that $p \neq q$. Let $\tau_1 = \{\emptyset\} \cup \{O \subseteqq X : p \in O\}$ and let $\tau_2 = \{X\} \cup \{O \subseteqq X : q \notin O\}$. If $\tau = \tau_1 \cap \tau_2$, then (X,τ) is strongly S-closed, since (X,τ_1) is strongly S-closed. Since $\ker\{q\} = X$, (X,τ) is Λ-compact. Clearly $\overline{\{x\}} = \{x,q\}$ for $x \neq p$, and thus $X \setminus \{p\}$ is a closed subspace that is not strongly S-closed. Hence (X,τ) fails to be LC-compact by Theorem 2.6.

Example 3.7 (a hereditarily compact, strongly S-closed space that is not LC-compact). Let $X = \mathbb{N}$. It is well known that there exists a family $\mathcal{A} = \{A_i : i \in I\}$ of distinct infinite subsets of $\mathbb{N}$, where $A_i \cap A_j$ is at most finite whenever $i \neq j$, and where I has the cardinality of the reals. In [7], p. 914, a topology τ_1 is defined by defining the closed sets to be X and all sets of the form $E \cup A_{i_1} \cup \cdots \cup A_{i_k}$ where E is finite. It is pointed out in [7] that (X,τ_1) is T_1 and hereditarily compact. Now pick a point $p \in X$ and $j \in I$ such that $p \notin A_j$. Let $\tau_2 = \{\emptyset\} \cup \{O \subseteqq X : p \in O\}$ and let $\tau = \tau_1 \cap \tau_2$. Clearly (X,τ) is hereditarily compact and strongly S-closed. Observe also that $\{x\}$ is closed in (X,τ) whenever $x \neq p$. Hence A_j is a closed T_1 subspace and thus cannot be strongly S-closed. By Theorem 2.6, (X,τ) fails to be LC-compact.

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