A DECOMPOSITION OF CONTINUITY

Maximilian GANSTER and Ivan REILLY

appeared in: Acta Math. Hungarica 56(3–4) (1990), 299–301.

In 1922 Blumberg[1] introduced the notion of a real valued function on Euclidean space being densely approached at a point in its domain. Continuous functions satisfy this condition at each point of their domains. This concept was generalized by Ptak[7] in 1958 who used the term 'nearly continuous', and by Husain[3] in 1966 under the name of 'almost continuity'. More recently, Mashhour et al. [5] have called this property of functions between arbitrary topological spaces 'precontinuity'.

In this paper we define a new property of functions between topological spaces which is the dual of Blumberg's original notion, in the sense that together they are equivalent to continuity. Thus we provide a new decomposition of continuity in Theorem 4 (iv) which is of some historical interest.

In a recent paper [10] , Tong introduced the notion of an $\mathcal{A}$–set in a topological space and the concept of $\mathcal{A}$–continuity of functions between topological spaces. This enabled him to produce a new decomposition of continuity. In this paper we improve Tong's decomposition result and provide a decomposition of $\mathcal{A}$–continuity.

Let S be a subset of a topological space (X, τ) . We denote the closure of S and the interior of S with respect to τ by clS and $intS$ respectively.

Definition 1 A subset S of (X,τ) is called

(i) an *α–set* if $S \subseteq int(cl(intS))$,

(ii) a *semiopen set* if $S \subseteq cl(intS)$,

(iii) a *preopen set* if $S \subseteq int(clS)$,

(iv) an *$\mathcal{A}$–set* if $S = U \cap F$ where U is open and F is regular closed,

(v) *locally closed* if $S = U \cap F$ where U is open and F is closed.

Recall that S is regular closed in (X,τ) if $S = cl(intS)$. We shall denote the collections of regular closed, locally closed, preopen and semiopen subsets of (X,τ) by $RC(X,\tau)$, $LC(X,\tau)$, $PO(X,\tau)$ and $SO(X,\tau)$ respectively. The collections of $\mathcal{A}$–sets in (X,τ) will be denoted by $\mathcal{A}(X,\tau)$. Following the notation of Njastad[6] , τ^α will denote the collection of all α–sets in (X,τ) .

The notions in Definition 1 were introduced by Njastad [6], Levine [4], Mashhour et al. [5], Tong [10] and Bourbaki [2] respectively. Stone [9] used them term FG for a locally closed subset. We note that a subset S of (X,τ) is locally closed iff $S = U \cap clS$ for some open set U ([2], I.3.3, Proposition 5).

Corresponding to the five concepts of generalized open set in Definition 1, we have five variations of continuity.

Definition 2 A function $f : X \to Y$ is called *α–continuous* (*semicontinuous, precontinuous, $\mathcal{A}$–continuous, LC–continuous* respectively) if the inverse image under f of each open set in Y is an α–set (semiopen, preopen, $\mathcal{A}$–set, locally closed respectively) in X.

Njastad [6] introduced α–continuity, Levine [4] semicontinuity and Tong [10] $\mathcal{A}$–continuity, while LC–continuity seems to be a new notion. It is clear that $\mathcal{A}$–continuity implies LC–continuity. We now provide an example to distinguish these concepts.

Example 1 Let (X,τ) be the set $\mathbb{N}$ of positive integers with the cofinite topology. Define the function $f : X \to X$ by $f(1) = 1$ and $f(x) = 2$ for all $x \neq 1$. Then $V = X \setminus \{2\}$ is open and $f^{-1}(V) = \{1\}$ which is (locally) closed but not an $\mathcal{A}$–set. Not that the only regular closed subsets of (X, τ) are $\emptyset$ and X . For any subset V of X , $f^{-1}(V)$ is $\{1\}$, $X \setminus \{1\}$, $\emptyset$ or X , and these are all locally closed subsets of X . Hence f is LC–continous but not $\mathcal{A}$–continuous.

Theorem 1 Let S be a subset of a topological space (X, τ) . Then S is an $\mathcal{A}$–set if and only if S is semiopen and locally closed.

Proof. Let $S \in \mathcal{A}(X, \tau)$, so $S = U \cap F$ where $U \in \tau$ and $F \in RC(X, \tau)$. Clearly S is locally closed. Now $intS = U \cap intF$, so that $S = U \cap cl(intF) \subseteq cl(U \cap intF) = cl(intS)$, and hence S is semiopen.

Conversely, let S be semiopen and locally closed, so that $S \subseteq cl(intS)$ and $S = U \cap clS$ where U is open. Then $clS = cl(intS)$ and so is regular closed. Hence S is an $\mathcal{A}$–set. □

Theorem 2 For a subset S of a topological space (X, τ) the following are equivalent:

(1) S is open.

(2) S is an α–set and locally closed.

(3) S is preopen and locally closed.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) : Let S be preopen and locally closed, so that $S \subseteq int(clS)$ and $S = U \cap clS$. Then $S \subseteq U \cap int(clS) = int(U \cap clS) = intS$, hence S is open. □

Theorem 3 For a topological space (X, τ) the following are equivalent:

(1) $\mathcal{A}(X, \tau) = \tau$.

(2) $\mathcal{A}(X, \tau)$ is a topology on X .

(3) The intersection of any two $\mathcal{A}$–sets in X is an $\mathcal{A}$–set.

(4) $SO(X, \tau)$ is a topology on X .

(5) (X, τ) is extremally disconnected.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4) : Let $S_1, S_2 \in SO(X,\tau)$. We wish to show $S_1 \cap S_2 \in SO(X,\tau)$. Suppose there is a point $x \in S_1 \cap S_2$ such that $x \notin cl(int(S_1 \cap S_2))$. So there is an open neighbourhood U of x such that $U \cap intS_1 \cap intS_2 = \emptyset$. Thus $U \cap clS_1 \cap intS_2 = \emptyset$ and hence we have $U \cap int(clS_1) \cap clS_2 = \emptyset$. Therefore $U \cap int(clS_1 \cap clS_2) = \emptyset$, so that $x \notin cl(int(clS_1 \cap clS_2))$. But, on the other hand we have $clS_1, clS_2 \in RC(X,\tau)$, so that $clS_1, clS_2 \in \mathcal{A}(X,\tau) \subseteq SO(X,\tau)$. Then $x \in clS_1 \cap clS_2$ implies $x \in cl(int(clS_1 \cap clS_2))$, which is a contradiction. Thus no such point x exists, and so $S_1 \cap S_2 \in SO(X,\tau)$.

(4) $\Rightarrow$ (5) : is due to Njastad [6] .

(5) $\Rightarrow$ (1) : If A is an $\mathcal{A}$–set then $A = U \cap F$ where $U \in \tau$ and $F \in RC(X,\tau)$. Since (X,τ) is extremally disconnected, $F \in \tau$. Hence $A \in \tau$. $\Box$

Theorem 1 and 2 show that in any topological space (X,τ) we have the following fundamental relationships between the classes of subsets of X we are considering, namely

(i) $\mathcal{A}(X,\tau) = SO(X,\tau) \cap LC(X,\tau)$.

(ii) $\tau = \tau^\alpha \cap LC(X,\tau)$.

(iii) $\tau = PO(X,\tau) \cap LC(X,\tau)$.

(iv) $\tau = PO(X,\tau) \cap \mathcal{A}(X,\tau)$.

(v) $\tau^\alpha = PO(X,\tau) \cap SO(X,\tau)$ (is due to Reilly and Vamanamurthy [8])

These relationships provide immediate proofs for the following decompositions. We note that (ii) of Theorem 4 is an improvement of Tong's decomposition of continuity [10], Theorem 4.1, and that (iii) of Theorem 4 is due to Reilly and Vamanamurthy [8] . Theorem 4 (i), (iv) and (v) seem to be new results and provide new decompositions of continuity.

Theorem 4 Let $f : X \to Y$ be a function. Then

(i) f is $\mathcal{A}$–continuous if and only if f is semicontinuous and LC–continuous.

(ii) f is continuous if and only if f is α–continuous and LC–continuous.

(iii) f is α–continuous if and only if f is precontinuous and semicontinuous.

(iv) f is continuous if and only if f is precontinuous and LC–continuous.

(v) f is continuous if and only if f is precontinuous and $\mathcal{A}$–continuous.

References

[1] H. Blumberg, *New properties of real functions*, Trans. Amer. Math. Soc. 24 (1922), 113–128.

[2] N. Bourbaki, *General Topology*, Part 1, Addison–Wesley (Reading, Mass. 1966).

[3] T. Husain, *Almost continuous mappings*, Prace Mat. 10 (1966), 1–7.

[4] N. Levine, *Semiopen sets and semicontinuity in topological spaces*, Amer. Math. Monthly 70 (1963), 36–41.

[5] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. and Phys. Soc. Egypt 51 (1981).

[6] O. Njastad, *On some classes of nearly open sets*, Pacific J. Math. 15 (1965), 961–970.

[7] V. Ptak, *Completenss and open mapping theorem*, Bull. Soc. Math. France 86 (1958), 41–74.

[8] I.L. Reilly and M.K. Vamanamurthy, *On α–continuity in topological spaces*, Acta Math. Hungar. 45 (1985), 27–32.

[9] A.H. Stone, *Absolutely FG spaces*, Proc. Amer. Math. Soc. 80 (1980), 515–520.

[10] J. Tong, *A decomposition of continuity*, Acta Math. Hungar. 48 (1986), 11–15.

Department of Mathematics, Graz University of Technology, Graz, AUSTRIA.

Department of Mathematics and Statistics, University of Auckland, Auckland, NEW ZEALAND.