A DECOMPOSITION OF CONTINUITY

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In 1922 Blumberg[1] introduced the notion of a real valued function on Euclidean space being densely approached at a point in its domain. Continuous functions satisfy this condition at each point of their domains. This concept was generalized by Ptak[7] in 1958 who used the term 'nearly continuous', and by Husain[3] in 1966 under the name of 'almost continuity'. More recently, Mashhour et al. [5] have called this property of functions between arbitrary topological spaces 'precontinuity'.

In this paper we define a new property of functions between topological spaces which is the dual of Blumberg’s original notion, in the sense that together they are equivalent to continuity. Thus we provide a new decomposition of continuity in Theorem 4 (iv) which is of some historical interest.

In a recent paper [10] , Tong introduced the notion of an $\mathcal{A}$–set in a topological space and the concept of $\mathcal{A}$–continuity of functions between topological spaces. This enabled him to produce a new decomposition of continuity. In this paper we improve Tong’s decomposition result and provide a decomposition of $\mathcal{A}$–continuity.

Let $S$ be a subset of a topological space $(X, \tau)$ . We denote the closure of $S$ and the interior of $S$ with respect to $\tau$ by $clS$ and $intS$ respectively.
**Definition 1** A subset $S$ of $(X, \tau)$ is called

(i) an $\alpha$–set if $S \subseteq \text{int}(\text{cl}(\text{int}S))$ ,

(ii) a semiopen set if $S \subseteq \text{cl}(\text{int}S)$ ,

(iii) a preopen set if $S \subseteq \text{int}(\text{cl}S)$ ,

(iv) an $\mathcal{A}$–set if $S = U \cap F$ where $U$ is open and $F$ is regular closed,

(v) locally closed if $S = U \cap F$ where $U$ is open and $F$ is closed.

Recall that $S$ is regular closed in $(X, \tau)$ if $S = \text{cl}(\text{int}S)$ . We shall denote the collections of regular closed, locally closed, preopen and semiopen subsets of $(X, \tau)$ by $\text{RC}(X, \tau)$, $\text{LC}(X, \tau)$, $\text{PO}(X, \tau)$ and $\text{SO}(X, \tau)$ respectively. The collections of $\mathcal{A}$–sets in $(X, \tau)$ will be denoted by $\mathcal{A}(X, \tau)$ . Following the notation of Njastad[6] , $\tau^\alpha$ will denote the collection of all $\alpha$–sets in $(X, \tau)$ .

The notions in Definition 1 were introduced by Njastad [6], Levine [4], Mashhour et al. [5], Tong [10] and Bourbaki [2] respectively. Stone [9] used them term $FG$ for a locally closed subset. We note that a subset $S$ of $(X, \tau)$ is locally closed iff $S = U \cap \text{cl}S$ for some open set $U$ ([2], I.3.3, Proposition 5).

Corresponding to the five concepts of generalized open set in Definition 1, we have five variations of continuity.

**Definition 2** A function $f : X \to Y$ is called $\alpha$–continuous (semicontinuous, precontinuous, $\mathcal{A}$–continuous, $\text{LC}$–continuous respectively) if the inverse image under $f$ of each open set in $Y$ is an $\alpha$–set (semiopen, preopen, $\mathcal{A}$–set, locally closed respectively) in $X$.

Njastad [6] introduced $\alpha$–continuity, Levine [4] semicontinuity and Tong [10] $\mathcal{A}$–continuity, while $\text{LC}$–continuity seems to be a new notion. It is clear that $\mathcal{A}$–continuity implies $\text{LC}$–continuity. We now provide an example to distinguish these concepts.

**Example 1** Let $(X, \tau)$ be the set $\mathbb{N}$ of positive integers with the cofinite topology. Define the function $f : X \to X$ by $f(1) = 1$ and $f(x) = 2$ for all $x \neq 1$ . Then $V = X \setminus \{2\}$ is open and $f^{-1}(V) = \{1\}$ which is (locally) closed but not an $\mathcal{A}$–set. Not that the only regular
closed subsets of $(X, \tau)$ are $\emptyset$ and $X$. For any subset $V$ of $X$, $f^{-1}(V)$ is $\{1\}$, $X \setminus \{1\}$, $\emptyset$ or $X$, and these are all locally closed subsets of $X$. Hence $f$ is $LC$–continuous but not $A$–continuous.

**Theorem 1** Let $S$ be a subset of a topological space $(X, \tau)$. Then $S$ is an $A$–set if and only if $S$ is semiopen and locally closed.

**Proof.** Let $S \in A(X, \tau)$, so $S = U \cap F$ where $U \in \tau$ and $F \in RC(X, \tau)$. Clearly $S$ is locally closed. Now $intS = U \cap int F$, so that $S = U \cap cl(int F) \subseteq cl(U \cap int F) = cl(int S)$, and hence $S$ is semiopen.

Conversely, let $S$ be semiopen and locally closed, so that $S \subseteq cl(int S)$ and $S = U \cap cl S$ where $U$ is open. Then $cl S = cl(int S)$ and so is regular closed. Hence $S$ is an $A$–set. □

**Theorem 2** For a subset $S$ of a topological space $(X, \tau)$ the following are equivalent:

1. $S$ is open.
2. $S$ is an $\alpha$–set and locally closed.
3. $S$ is preopen and locally closed.

**Proof.** (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) : Let $S$ be preopen and locally closed, so that $S \subseteq int(cl S)$ and $S = U \cap cl S$. Then $S \subseteq U \cap int(cl S) = int(U \cap cl S) = int S$, hence $S$ is open. □

**Theorem 3** For a topological space $(X, \tau)$ the following are equivalent:

1. $A(X, \tau) = \tau$.
2. $A(X, \tau)$ is a topology on $X$.
3. The intersection of any two $A$–sets in $X$ is an $A$–set.
4. $SO(X, \tau)$ is a topology on $X$.
5. $(X, \tau)$ is extremally disconnected.
Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear.

(3) ⇒ (4) : Let $S_1, S_2 \in SO(X, \tau)$ . We wish to show $S_1 \cap S_2 \in SO(X, \tau)$ . Suppose there is a point $x \in S_1 \cap S_2$ such that $x \notin cl(int(S_1 \cap S_2))$ . So there is an open neighbourhood $U$ of $x$ such that $U \cap int(S_1 \cap intS_2) = \emptyset$ . Thus $U \cap clS_1 \cap intS_2 = \emptyset$ and hence we have $U \cap int(clS_1 \cap clS_2) = \emptyset$ . Therefore $U \cap int(clS_1 \cap clS_2) = \emptyset$ , so that $x \notin cl(int(clS_1 \cap clS_2))$ .

But, on the other hand we have $clS_1, clS_2 \in RC(X, \tau)$ , so that $clS_1, clS_2 \in A(X, \tau) \subseteq SO(X, \tau)$ . Then $x \in clS_1 \cap clS_2$ implies $x \in cl(int(clS_1 \cap clS_2))$ , which is a contradiction. Thus no such point $x$ exists, and so $S_1 \cap S_2 \in SO(X, \tau)$ .


(5) ⇒ (1) : If $A$ is an $A$–set then $A = U \cap F$ where $U \in \tau$ and $F \in RC(X, \tau)$ . Since $(X, \tau)$ is extremally disconnected, $F \in \tau$ . Hence $A \in \tau$ . □

Theorem 1 and 2 show that in any topological space $(X, \tau)$ we have the following fundamental relationships between the classes of subsets of $X$ we are considering, namely

(i) $A(X, \tau) = SO(X, \tau) \cap LC(X, \tau)$ .

(ii) $\tau = \tau^\alpha \cap LC(X, \tau)$ .

(iii) $\tau = PO(X, \tau) \cap LC(X, \tau)$ .

(iv) $\tau = PO(X, \tau) \cap A(X, \tau)$ .

(v) $\tau^\alpha = PO(X, \tau) \cap SO(X, \tau)$ (is due to Reilly and Vamanamurthy [8])

These relationships provide immediate proofs for the following decompositions. We note that (ii) of Theorem 4 is an improvement of Tong’s decomposition of continuity [10], Theorem 4.1, and that (iii) of Theorem 4 is due to Reilly and Vamanamurthy [8] . Theorem 4 (i), (iv) and (v) seem to be new results and provide new decompositions of continuity.

**Theorem 4** Let $f : X \rightarrow Y$ be a function. Then

(i) $f$ is $A$–continuous if and only if $f$ is semicontinuous and $LC$–continuous.

(ii) $f$ is continuous if and only if $f$ is $\alpha$–continuous and $LC$–continuous.

(iii) $f$ is $\alpha$–continuous if and only if $f$ is precontinuous and semicontinuous.

(iv) $f$ is continuous if and only if $f$ is precontinuous and $LC$–continuous.

(v) $f$ is continuous if and only if $f$ is precontinuous and $A$–continuous.
References


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