REMARKS ON LOCALLY CLOSED SETS *

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Abstract

This paper provides a useful characterization of $LC(X, \tau^\alpha)$, i.e. the family of locally closed subsets of (X, τ^α), where τ^α denotes the α–topology of a given topological space (X, τ) . In addition, we consider various statements about the family of locally closed subsets of an arbitrary space and examine the relationships between these statements.

1 Introduction and preliminaries

Recently there has been some interest in the notion of a locally closed subset of a topological space. According to Bourbaki [4] a subset S of a space (X, τ) is called *locally closed* if it is the intersection of an open set and a closed set. Ganster and Reilly used locally closed sets in [7] and [8] to define the concept of LC–continuity, i.e. a function $f : (X, \tau) \to (Y, \sigma)$ is *LC–continuous* if the inverse with respect to f of any open set in Y is locally closed in X . This enabled them to produce a decomposition of continuity for functions between arbitrary topological spaces. Later on, Jelic [9] extended their results to the bitopological setting by providing a decomposition of pairwise continuity and, quite recently, Balachandran and Sundaram studied several variations of LC-continuity in [5] and [13] . Finally, locally closed sets have been used by Aho and Nieminen [1] in their study of α–spaces and irresolvability.

In this paper we begin by characterizing $LC(X, \tau^\alpha)$, i.e. the family of locally closed subsets of (X, τ^α) where τ^α denotes the associated α–topology of a space (X, τ) . Our first result points out that the family (X, τ^α) has already been investigated by Kuratowski [10] in a different context and, moreover, that $LC(X, \tau^\alpha)$ coincides with the collection of δ–sets in (X, τ) [6] . We then move on to consider various statements about the family of locally closed subsets of an arbitrary space (X, τ) and examine the relationships between these statements.

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Let (X,τ) be a topological space. For a subset S of X , the closure and the interior of S with respect to (X,τ) will be denoted by clS and $intS$, respectively.

Definition 1 A subset S of a space (X,τ) is called
i) *semi-open* if $S \subseteq cl(intS)$,
ii) *semi-closed* if $X \setminus S$ is semi-open, or, equivalently, if $int(clS) \subseteq S$,
iii) an α–set if $S \subseteq int(cl(intS))$,
iv) *nwd* (=nowhere dense) if $int(clS) = \emptyset$.

The collections of semi-open sets, semi-closed sets and α–sets in (X,τ) will be denoted by $SO(X,\tau)$, $SC(X,\tau)$ and τ^α , respectively. Njastad [11] has shown that τ^α is a topology on X with the following properties : $\tau \subseteq \tau^\alpha$, $(\tau^\alpha)^\alpha = \tau^\alpha$ and $S \in \tau^\alpha$ if and only if $S = U \setminus N$ where $U \in \tau$ and N is nwd in (X,τ) . Hence $\tau = \tau^\alpha$ if and only if every nwd set in (X,τ) is closed. Clearly every α–set is semi-open and every nwd set in (X,τ) is semi-closed. Andrijevic [2] has observed that $SO(X,\tau^\alpha) = SO(X,\tau)$, and that $N \subseteq X$ is nwd in (X,τ^α) if and only if N is nwd in (X,τ) .

Definition 2 A subset S of (X,τ) is called
i) *locally closed* if $S = U \cap F$ where U is open and F is closed, or, equivalently, if $S = U \cap clS$ for some open set U .
ii) *co-locally closed* if $X \setminus S$ is locally closed, or, equivalently, if $S = U \cup F$ where U is open and F is closed and nwd.

We will denote the collections of all locally closed sets and co-locally closed sets of (X,τ) by $LC(X,\tau)$ and co-$LC(X,\tau)$, respectively. Note that Stone [12] has used the term FG for a locally closed subset. A dense subset of (X,τ) is locally closed if and only if it is open. More generally, Ganster and Reilly [7] have pointed out that, if $S \subseteq X$ is nearly open, i.e. if $S \subseteq int(clS)$, then S is locally closed if and only if S is open. It is easy to check that (X,τ) is submaximal, i.e. every dense set is open, if and only if every subset of X is locally closed. Finally, spaces in which singletons are locally closed are called T_D-spaces [3] .

No separation axioms are assumed unless explicitly stated.

2 Locally closed sets in α-spaces

Let (X,τ) be a topological space and let us denote by $\mathcal{J}$ the ideal of nwd subsets of (X,τ) . On page 69 in [10] , Kuratowski defined a subset $A \subseteq X$ to be open *mod* $\mathcal{J}$ if there exists an open set G such that $A \setminus G \in \mathcal{J}$ and $G \setminus A \in \mathcal{J}$.

Proposition 2.1 (see page 69 in [10])

Let $\mathcal{J}$ denote the ideal of nwd sets in a space (X, τ) . Then

1) open sets are open *mod* $\mathcal{J}$,
2) closed sets are open *mod* $\mathcal{J}$,
3) if A, B are open *mod* $\mathcal{J}$, then $A \cap B$, $A \cup B$ and $X \setminus A$ are open *mod* $\mathcal{J}$,
4) $A \subseteq X$ is open *mod* $\mathcal{J}$ if and only if $A = U \cup N$ where U is open and N is nwd in (X, τ) .

In order to state our main result in this section we need some more definitions. A subset S of a space (X, τ) is called *semi-locally closed* [13] if it is the intersection of a semi-open set and a semi-closed set. $S \subseteq X$ is said to be a δ*–set* in (X, τ) [6] if $int(clS) \subseteq cl(intS)$.

Theorem 2.2 Let A be a subset of a space (X, τ) and let $\mathcal{J}$ denote the ideal of nwd subsets of (X, τ) . Then the following are equivalent :

1) $A \in LC(X, \tau^\alpha)$,
2) A is semi-locally closed ,
3) A is a δ–set ,
4) $A = U \cup N$ where U is open and N is nwd in (X, τ) ,
5) A is open mod $\mathcal{J}$.

Proof. 1) $\Rightarrow$ 2) : This is obvious since every α–set is semi-open.

2) $\Rightarrow$ 3) : Let $A = S \cap T$ where $S \in SO(X, \tau)$ and $T \in SC(X, \tau)$, i.e. $S \subseteq cl(intS)$ and $int(clT) \subseteq T$. Since $int(clA) \subseteq int(clT) \subseteq T$, we have $int(clA) \subseteq intT$. Since $A \subseteq S \subseteq cl(intS)$ we have $int(clA) \subseteq cl(intS)$. Consequently, $int(clA) \subseteq cl(intS) \cap intT \subseteq cl(intS \cap intT) = cl(intA)$. Hence A is a δ–set.

3) $\Rightarrow$ 4) : Suppose that $int(clA) \subseteq cl(intA)$ and let $U = intA$ and $N = A \setminus intA$. We will show that N is nwd . Clearly $int(clN) \subseteq int(clA)$, and since $N \cap intA = \emptyset$, we have $int(clN) \cap cl(intA) = \emptyset$. So $int(clN) = \emptyset$, i.e. N is nwd.

4) $\Leftrightarrow$ 5) : See Proposition 2.1 .

5) $\Rightarrow$ 1) : Let A be open mod $\mathcal{J}$. By Proposition 2.1 , $X \setminus A$ is open mod $\mathcal{J}$, so $X \setminus A = U \cup N$ where $U \in \tau$ and N is nwd in (X, τ) . Hence $A = (X \setminus N) \cap (X \setminus U) \in LC(X, \tau^\alpha)$ since $X \setminus N \in \tau^\alpha$ and $X \setminus U$ is closed in (X, τ) and thus closed in (X, τ^α) . $\Box$

Corollary 2.3 $SO(X, \tau) \subseteq LC(X, \tau^\alpha)$ and $SC(X, \tau) \subseteq LC(X, \tau^\alpha)$ for every space (X, τ) .

Corollary 2.4 If $f : (X, \tau) \to (Y, \sigma)$ is quasi-continuous, i.e. the inverse image of every open set is semi-open, then $f : (X, \tau^\alpha) \to (Y, \sigma)$ is LC-continuous.

Remark 2.5 In a recent paper [6] , Chattopadhyay and Bandyopadhyay study the collection T^δ of all δ–sets of a space (X,τ) . Using Theorem 2.2 one obtains straightforward proofs of many results in [6] , e.g.

1) T^δ is the discrete topology if and only if (X,τ^α) is submaximal (since 1) $\Leftrightarrow$ 3) in Theorem 2.2) .

2) $(\tau^\alpha)^\delta = T^\delta$ (since $(\tau^\alpha)^\alpha = \tau^\alpha$) .

3) $\tau = T^\delta$ if and only if every open set is closed.

Finally let us observe that (X,τ^α) is a T_D space if and only if every singleton is a δ–set in (X,τ).

3 On the structure of $LC(X,\tau)$

The topic of this section is the relationship between the following properties of a space (X,τ) :

(A) $SO(X,\tau) \subseteq LC(X,\tau)$;

(B) co-$LC(X,\tau) \subseteq LC(X,\tau)$;

(C) $SC(X,\tau) \subseteq LC(X,\tau)$;

(D) Every nwd set in (X,τ) is locally closed in (X,τ) .

Theorem 3.1 For a space (X,τ) the following are equivalent :

1) (X,τ) satisfies (A) ,

2) $\tau = \tau^\alpha$,

3) $LC(X,\tau) = LC(X,\tau^\alpha)$.

Proof. 1) $\Rightarrow$ 2) : Let N be nwd in (X,τ) . Then $X \setminus N$ is dense and semi-open in (X,τ) , hence, by assumption, locally closed. Thus $X \setminus N \in \tau$ and so N is closed in (X,τ) . Hence $\tau = \tau^\alpha$.

2) $\Rightarrow$ 3) : This is obvious.

3) $\Rightarrow$ 1) : Let $S \in SO(X,\tau)$. By Corollary 2.3 we have $S \in LC(X,\tau^\alpha)$. Thus $S \in LC(X,\tau)$ and so (X,τ) satisfies (A) . $\square$

Corollary 3.2 For every space (X,τ) , (X,τ^α) satisfies (A) .

Our next result follows immediately from Proposition 2.1 and Theorem 3.1

Theorem 3.3 For a space (X,τ) the following holds :

1) (A) implies (B) ,

2) (A) implies (C) implies (D) .

We now provide examples to show that none of the implications in Theorem 3.3 can be reversed.

Example 3.4 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $LC(X, \tau) = \{\emptyset, \{a\}, \{b, c\}, X\} = co - LC(X, \tau)$. Hence (X, τ) satisfies (B), but fails to satisfy (A) since $\{a, c\} \in SO(X, \tau) \setminus LC(X, \tau)$.

Example 3.5 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then we have $LC(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$ and $SC(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Hence (X, τ) satisfies (C) but not (A) since $\{a, c\} \in SO(X, \tau) \setminus LC(X, \tau)$.

Example 3.6 Let $X = \{a, b, x, y, z\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, x\}, \{a, b, x, y\}, \{a, b, x, z\}, X\}$. If $Z = \{x, y, z\}$ then Z is a closed subspace of (X, τ) and the subspace topology $\tau|Z = \{\emptyset, \{x\}, \{x, y\}, \{x, z\}, Z\}$ is submaximal. Now, if $N \subseteq X$ is nwd in (X, τ) then $N \subseteq Z$ and $N \in LC(Z, \tau|Z)$ and, since Z is closed in (X, τ) , $N \in LC(X, \tau)$. Hence (X, τ) satisfies (D) .

On the other hand, if $A = \{a, y\}$ then $A \in SC(X, \tau)$. We have, however, $x \in U \cap clA$ for any open set U containing y , so $A \notin LC(X, \tau)$. Hence (X, τ) does not satisfy (C) .

In order to state and prove our final result let us say that a space (X, τ) is a T_1^* space if every nwd subset is a union of closed sets. Clearly every T_1 space is a T_1^* space , and the indiscrete topology on any set X having at least two points yields a T_1^* space which is not T_1 .

Theorem 3.7 For a space (X, τ) the following are equivalent :

1) (X, τ) satisfies (A) ,
2) (X, τ) satisfies (B) and (D) ,
3) (X, τ) is T_1^* and satisfies (B) .

Proof. 1) $\Rightarrow$ 2) : See Theorem 3.3 .

2) $\Rightarrow$ 3) : Let N be nwd in (X, τ) and let $x \in N$. Then $\{x\} \in LC(X, \tau)$ by (D) . Since (B) holds, $X \setminus \{x\}$ is locally closed and dense, and so open. Thus $\{x\}$ is closed, and hence (X, τ) is T_1^* .

3) $\Rightarrow$ 1) : By Theorem 3.1 we have to show that every nwd subset N of (X, τ) is closed. Let $x \in clN$. Since (X, τ) is T_1^* and clN is nwd , $\{x\}$ is closed and so $clN \cap (X \setminus \{x\}) \in LC(X, \tau)$. Since (B) holds , $\{x\} \cup (X \setminus clN)$ is locally closed and dense, hence an open neighborhood of x . Consequently $N \cap (\{x\} \cup (X \setminus clN))$ is nonempty and so $x \in N$. Thus N is closed. $\square$

Corollary 3.8 In general, the statements (B) and (C) are independent of each other.

Corollary 3.9 [1] Let (X, τ) be a T_D space satisfying (B) . Then (X, τ) satisfies (A).

Proof. It is easy to show that (X, τ) is T_1^* . Now apply Theorem 3.3 $\Box$

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