S-SETS AND CO-S-CLOSED TOPOLOGIES

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Abstract

We consider the family of S-sets in a topological space (X,τ) and discuss the associated co-S-closed topology τ^* of (X,τ) . It is shown that the co-S-closed topology may be used to characterize the S-closedness of the space in question. In addition, we study the relationship between the semi-regularization topology, the co-S-closed topology and the given topology of a space.

1 Introduction

In 1963, Levine [8] introduced and studied the concept of semi-open sets in topological spaces. In 1976, Thompson [13] used semi-open sets to define the class of S-closed spaces. In his own study of S-closed spaces, Noiri [9] investigated certain subsets of a given topological space (X,τ) which he called S-closed relative to (X,τ) . Di Maio [1] , calling such subsets S-sets, observed that in any space (X,τ) the family of open sets whose complements are S-sets forms a base for a coarser topology τ^* on X . In [1] this topology was used to define and study so-called S-continuous functions between topological spaces. Quite recently, Jiang and Reilly [6] extended Di Maio's work on S-continuity, and they called the associated topology τ^* the co-S-closed topology of the space (X,τ) .

The purpose of this paper is to discuss in greater detail the concept of co-S-closed topologies. We will start with various results about S-sets in topological spaces and then move on to consider the co-S-closed topology for an arbitrary space. As one might expect, the co-S-closed topology will be particularly useful in characterizing S-closed spaces. One of our main results says that a space (X, τ) is S-closed if and only if τ_s, the semi-regularization of τ , is coarser than the co-S-closed topology τ^* . This result not only has some interesting consequences, it enables us also to provide very simple proofs for several known results. In the last section of our paper we will consider the relationship between co-S-closed topologies and the class of SC-compact spaces due to Garg and Sivaraj [3] . We will close by posing two open problems.

2 Preliminaries

No separation axioms are assumed unless explicitly stated. For a subset S of a topological space (X, τ) the closure and the interior of S with respect to (X, τ) will be denoted by $\tau - clS$ and $\tau - intS$ respectively. We will, however, usually suppress the τ when there is no possibility of confusion.

Definition 1 A subset S of a space (X, τ) is called

i) semi-open [8] if $S \subseteq cl(intS)$,

ii) regular open if $S = int(clS)$,

iii) regular closed if $X \setminus S$ is regular open, or equivalently, if $S = cl(intS)$.

The families of regular open subsets and regular closed subsets of a space (X, τ) are denoted by $RO(X, \tau)$ and $RC(X, \tau)$, respectively. Clearly, $S \in RO(X, \tau)$ if and only if S is the interior of some closed set, and $S \in RC(X, \tau)$ if and only if S is the closure of some open set. Since $RO(X, \tau)$ is closed under forming finite intersections, it constitutes a base for a coarser topology τ_s on X , and the space (X, τ_s) is called the *semi-regularization* of (X, τ) . A space (X, τ) is called *semi-regular* if $\tau = \tau_s$. It is well known that $(\tau_s)_s = \tau_s$ and thus (X, τ_s) is semi-regular. We call a space (X, τ) *hyperconnected* whenever any pair of nonempty open sets has nonempty intersection, or equivalently, whenever any nonempty open set is dense in

(X,τ) . Clearly (X,τ) is hyperconnected if and only if $\tau_s = RO(X,\tau) = \{\emptyset, X\}$. A space (X,τ) is said to be *extremally disconnected*, briefly e.d. , if the closure of any open set is open. It is obvious that (X,τ) is e.d. if and only if $RO(X,\tau) = RC(X,\tau)$. Finally, as a weaker form of Hausdorffness, Soundararajan [11] defined a space (X,τ) to be *weakly* T_2 if $\{x\} = \bigcap\}F \in RC(X,\tau) \ : \ x \in F\}$ for every $x \in X$.

Definition 2 A space (X,τ) is called *S-closed* [13] (respectively *quasi-H-closed*) if every cover of X by semi-open sets (respectively open sets) contains a finite subfamily the closures of whose members cover X .

Observation 2.1 A space (X,τ) is S-closed if and only if every cover of X by regular closed sets contains a finite subcover.

For convenience we would like to mention the following two results. The first one is a folklore result while the second one is due to Hermann [4] .

Proposition 2.2 Let τ and σ be topologies on a set X such that $\tau_s \subseteq \sigma \subseteq \tau$. Then $RO(X,\tau) = RO(X,\sigma)$ and $RO(X,\tau) = RO(X,\sigma)$, and thus $\sigma_s = \tau_s$.

Proposition 2.3 [4] Let (X,τ) be weakly-T_2 . Then (X,τ) is S-closed if and only if (X,τ) is quasi-H-closed and e.d. . Moreover, a weakly-T_2 S-closed space is T_2 .

3 S-sets in topological spaces

Definition 3 A subset S of a space (X,τ) is called an *S-set* in (X,τ) if every cover of S by regular closed sets of (X,τ) contains a finite subcover of S .

Note that Noiri [9] used the term 'S-closed relative to (X,τ)' . Obviously, (X,τ) is S-closed if and only if X is an S-set in (X,τ) . Moreover, $S \subseteq X$ is an S-set in (X,τ) if and only if S is an S-set in (X,τ_s) . Spaces in which every closed subset is an S-set have been called *SC-compact* [3] .

The following result of Noiri is fundamental in dealing with S-sets.

Proposition 3.1 [9] Let S be an S-set in (X, τ) . Then clS and $int(clS)$ are S-sets, and $S \cap G$ is an S-set whenever $G \in RO(X, \tau)$.

The proof of the preceding result reveals also that whenever $S \subseteq X$ is an S-set and $\{G_i : i \in I\} \subseteq RO(X, \tau)$ then $S \cap A$ is an S-set where $A = \bigcap\{G_i : i \in I\}$. As a consequence, if (X, τ) is S-closed then any intersection of regular open sets is an S-set in (X, τ) .

Our next observation points out that S-sets may be utilized to characterize weakly-T_2 spaces.

Proposition 3.2 For a space (X, τ) the following are equivalent :

1) (X, τ) is weakly-T_2 ,
2) (X, τ_s) is T_1 ,
3) Every S-set in (X, τ) is closed in (X, τ_s) .

Proof.

1) $\Leftrightarrow$ 2) : This is obvious.

2) $\Rightarrow$ 3) : Let S be an S-set in (X, τ) and let $x \notin S$. Since (X, τ_s) is T_1 , for every $y \in S$ there exists $F_y \in RC(X, \tau)$ such that $y \in F_y$ and $x \notin F_y$. Since S is covered by $\{F_y : y \in S\}$, there is a finite subfamily whose union contains S . If F denotes this union then $F \in RC(X, \tau)$ and $X \setminus F$ is a τ_s-open neighborhood of x having empty intersection with S . Hence S is τ_s-closed.

3) $\Rightarrow$ 2) : Observe that $\{x\}$ is an S-set in (X, τ) for every $x \in X$. $\square$

In his fundamental paper on hereditarily compact spaces, Stone [12] defined a space (X, τ) to be *semi-irreducible* if every family consisting of nonempty pairwise disjoint open sets has to be finite. Moreover, in [12] it is shown that (X, τ) is semi-irreducible if and only if $RO(X, \tau)$ is finite.

The following result shows that the semi-irreducibility of a space may also be characterized in terms of S-sets.

Theorem 3.3 A space (X, τ) is semi-irreducible if and only if every subset of X is an S-set.

Proof. If (X, τ) is semi-irreducible, i.e. if $RO(X, \tau)$ and thus $RC(X, \tau)$ are finite, then every subset of X is clearly an S-set. To prove the converse, suppose that every subset of X is an S-set in (X, τ) . Let us assume that $RC(X, \tau)$ is infinite. Since $RC(X, \tau)$ is a Boolean algebra, there exists a strictly increasing sequence $F_0 \subset F_1 \subset F_2 \subset ...$ in $RC(X, \tau)$ (see e.g. [7] , page 40). Let ω denote the set of natural numbers. For each $n \in \omega$ pick $x_n \in F_{n+1} \setminus F_n$ and let $S = \{x_n \ : \ n \in \omega\}$. By construction , S fails to be an S-set and thus we have arrived at a contradiction. Hence $RC(X, \tau)$ is finite, i.e. (X, τ) is semi-irreducible. □

In contrast to the preceding result we now address the question to describe spaces in which every S-set has to be finite. While we have not been able to completely characterize this class of spaces we do have the following result.

Theorem 3.4 Let (X, τ) be a regular T_1 space in which every singleton is the intersection of two regular closed sets. Then every S-set in (X, τ) is finite.

Proof. Let $S \subseteq X$ be an S-set in (X,τ) . For each $x \in S$, $X \setminus \{x\} = G_x \cup H_x$ where $G_x, H_x \in RO(X, \tau)$. By Proposition 3.1 , $S \cap G_x$ and $S \cap H_x$ are S-sets, and so $S \setminus \{x\}$ is an S-set for each $x \in S$. By Proposition 3.2, $S \setminus \{x\}$ is closed in (X, τ) for each $x \in S$. Since (X, τ) is regular, for each $x \in S$ there exists $V_x \in \tau$ containing x such that $clV_x \cap (S \setminus \{x\}) = \emptyset$, i.e. $clV_x \cap S = \{x\}$. Now S is an S-set and $\{clV_x \ : \ x \in S\}$ is a cover of S by regular closed sets, hence S must be finite. □

Corollary 3.5 The set of real numbers with the euclidean topology is a space in which every S-set is finite.

In concluding this section recall that Ganster [2] has defined a space to be *strongly s-regular* if every open set is the union of regular closed sets. The following observation is easily proved.

Proposition 3.6 If (X, τ) is strongly s-regular then every S-set in (X, τ) is compact.

4 Co-S-closed topologies

Di Maio [1] has observed that for any space (X, τ) the family $\{U \in \tau \ : \ X \setminus U$ is an S-set in $(X, \tau)\ \}$ is a base for a coarser topology τ^* on X which we will call the *co-S-closed topology* of (X, τ) . A basic result about the co-S-closed topology is the following.

Proposition 4.1 [1] Let τ^* be the co-S-closed topology of (X, τ) . If (X, τ^*) is not hyperconnected, then (X, τ) is S-closed.

Note that if X denotes the set of real numbers and τ the euclidean topology on X , then (X, τ) is not S-closed hence (X, τ^*) is hyperconnected. In fact,τ^* is the cofinite topology on X by Corollary 3.5 .

We are now able to state and prove one of our main results in this section.

Theorem 4.2 Let τ^* be the co-S-closed topology of a space (X, τ) . Then the following are equivalent :

1) (X, τ) is S-closed ,
2) $\tau_s \subseteq \tau^*$,
3) $RC(X, \tau^*) = RC(X, \tau)$.

Proof. 1) $\Rightarrow$ 2) : Let $G \in RO(X, \tau)$ and let $F = X \subseteq G$. Since X is an S-set and $intF \in RO(X, \tau)$, by Proposition 3.1 , $intF$ and $F = cl(intF)$ are S-sets and thus we have $G \in \tau^*$. Consequently, $\tau_s \subseteq \tau^*$.

2) $\Rightarrow$ 3) : Since $\tau_s \subseteq \tau^* \subseteq \tau$, by Proposition 2.2 we have $RC(X, \tau^*) = RC(X, \tau)$.

3) $\Rightarrow$ 1) : By Propositon 4.1 , if (X, τ^*) is not hyperconnected, then (X, τ) is S-closed. If (X, τ^*) is hyperconnected, then, by assumption, (X, τ) has to be hyperconnected and thus S-closed. □

The preceding result has a number of interesting consequences whose proofs are easy and hence left to the reader (just consider the two cases whether (X, τ^*) is hyperconnected or not, and apply Proposition 4.1 and Theorem 4.2).

Corollary 4.3 For any space (X, τ) , we always have $RC(X, \tau^*) \subseteq RC(X, \tau)$.

Corollary 4.4 For any space (X, τ) , (X, τ^*) is always S-closed .

Corollary 4.5 A space (X, τ) is S-closed if and only if (X, τ_s) is S-closed if and only if $\tau_s = (\tau_s)^*$.

Motivated by Theorem 4.2 , the question now arises under what conditions on a space (X, τ) the inclusion "$\tau^* \subseteq \tau_s$" holds. It turns out that a suitable weakening of the property "weakly-T_2" will do the job.

Definition 4 A space (X, τ) is said to be *subweakly* T_2 if $\tau - cl\{x\} = \tau_s - cl\{x\}$ for every $x \in X$.

Note that $\tau_s - cl\{x\} = \bigcap\{F \in RC(X, \tau) \ : \ x \in F\}$. Obviously every semi-regular space is subweakly-T_2 , and (X, τ) is weakly-T_2 if and only if (X, τ) is subweakly-T_2 and T_1 .

Observe also that (X, τ) is subweakly-T_2 and hyperconnected if and only if τ is the indiscrete topology on X . In particular, the cofinite topology on an infinite set yields a space which is T_1 but not subweakly-T_2 .

Theorem 4.6 Let τ^* be the co-S-closed topology of (X, τ) . Then (X, τ) is subweakly-T_2 if and only if $\tau^* \subseteq \tau_s$.

Proof. Suppose that (X, τ) is subweakly-T_2 . Let $U \in \tau$ such that $X \setminus U$ is an S-set in (X, τ) , and pick $x \in U$. We have to show that there exists $G \in RO(X, \tau))$ such that $x \in G \subseteq U$. For every $y \in X \setminus U$ we have $\tau - cl\{y\} \subseteq X \setminus U$ and so there exists $F_y \in RC(X, \tau)$ containing y but not x . Since $X \setminus U$ is an S-set, a finite subfamily of $\{F_y \ : \ y \in X \setminus U\}$ covers $X \setminus U$. If F denotes the union of this finite subfamily then $F \in RC(X, \tau)$. Hence $X \setminus F \in RO(X, \tau)$ and $x \in X \setminus F \subseteq U$. This shows that $\tau^* \subseteq \tau_s$.

Now suppose that $\tau^* \subseteq \tau_s$. If $x \in X$ then $\tau - cl\{x\}$ is an S-set and hence τ^*-closed. By assumption, $\tau - cl\{x\}$ is τ_s-closed and so $\tau - cl\{x\} = \tau_s - cl\{x\}$, i.e. (X, τ) is subweakly-T_2 . $\square$

As an immediate consequence of Theorem 4.2 and Theorem 4.6 we now have

Corollary 4.7 Let τ^* be the co-S-closed topology of the subweakly-T_2 space (X, τ) .

Then (X, τ) is S-closed if and only if $\tau^* = \tau_s$.

Corollary 4.8 [6] Let τ^* be the co-S-closed topology of (X, τ) . Then (X, τ^*) is weakly-T_2 if and only if (X, τ) is S-closed and T_2 .

Proof. If (X, τ^*) is weakly-T_2 then it is not hyperconnected and thus (X, τ) is S-closed by Proposition 4.1 . By Theorem 4.2 we have that (X, τ) is weakly-T_2 , and a weakly-T_2 S-closed space is T_2 by Proposition 2.3 .

Conversely, if (X, τ) is T_2 and S-closed , then (X, τ_s) is T_2 and $\tau^* = \tau_s$ by Corollary 4.7 , hence (X, τ^*) is clearly weakly-T_2 . □

In dealing with S-closed spaces (X, τ) , we have the given topology τ , the semi-regularization topology τ_s , and the co-S-closed topology τ^* of (X, τ) . The relationship between these topologies is $\tau_s \subseteq \tau^* \subseteq \tau$. Note that if X is an infinite set and τ is the cofinite topology on X then $\tau^* = \tau$ and $\tau_s \neq \tau^*$. On the other hand, if (X, τ) denotes the Katetov extension of the natural numbers then (X, τ) is T_2 and S-closed but not semi-regular (see e.g. [10]) . Hence $\tau_s = \tau^*$ and $\tau^* \neq \tau$. These observations lead to the question of describing the class of spaces (X, τ) which satisfy $\tau = \tau^*$. Clearly such spaces have to be S-closed by Theorem 4.2 . It is also obvious that SC-compact spaces (X, τ) [3] , i.e. spaces in which every closed set is an S-set, and semi-irreducible spaces satisfy $\tau = \tau^*$. While we did not succeed in characterizing the class of spaces (X, τ) which satisfy $\tau = \tau^*$,we do have some partial results. The first one is a straightforward consequence of Corollary 4.7 and Theorem 4.6 .

Theorem 4.9 Let τ^* be the co-S-closed topology of (X, τ) . Then (X, τ) is S-closed and semi-regular if and only if (X, τ) is subweakly-T_2 and $\tau = \tau^*$.

Recall that a space (X, τ) is said to be R_0 if $\tau - cl\{x\} \subseteq U$ whenever $U \in \tau$ and $x \in U$. Jankovic and Konstadilaki [5] have shown that an S-closed space (X, τ) is e.d. if and only if (X, τ_s) is R_0 . This result can also be stated in the following form.

Proposition 4.10 A space (X,τ) is quasi-H-closed and e.d. if and only if (X,τ) is S-closed and (X,τ_s) is R_0 .

As a consequence of Proposition 4.10 we now have

Theorem 4.11 Let τ^* be the co-S-closed topology of (X,τ) . If (X,τ) is subweakly-T_2 and (X,τ_s) is R_0 , then $\tau = \tau^*$ if and only if (X,τ) is SC-compact.

Proof. If (X,τ) is SC-compact then clearly $\tau = \tau^*$. Now suppose that $\tau = \tau^*$. Then (X,τ) is S-closed and $\tau = \tau^* = \tau_s$ by Theorem 4.6 . If $A \subseteq X$ is τ-closed then $A = \bigcap\{F \in RC(X,\tau) \ : \ A \subseteq F\}$. By Proposition 4.10 , (X,τ) is e.d. and so A is an intersection of regular open sets in (X,τ) and thus an S-set in (X,τ) . This proves that (X,τ) is SC-compact. $\square$

Corollary 4.12 Let (X,τ) be weakly-T_2 . Then $\tau = \tau^*$ if and only if (X,τ) is SC-compact.

In closing this paper we would like to pose the following two open problems.

Problem 1. Characterize the class of spaces in which every S-set is finite.

Problem 2. Let τ^* denote the co-S-closed topology of (X,τ) . Does $\tau = \tau^*$ imply in general that (X,τ) has to be SC-compact ? If the answer is "no", what is a useful condition (P) such that the following holds : " $\tau = \tau^*$ if and only if (X,τ) is S-closed and satisfies (P)" ?

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