COUNTABLY S-CLOSED SPACES *

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Abstract

In this paper we introduce the class of countably S-closed spaces which lies between the familiar classes of S-closed spaces and feebly compact spaces. We characterize countably S-closed spaces and study their basic properties. In addition, we investigate the relationship between countably S-closed spaces and feebly compact spaces. Several examples illustrate our results.

1 Introduction and Preliminaries

In 1976, Thompson [8] introduced the class of S-closed spaces. A space X is called Sclosed if every semi-open cover has a finite subfamily the closures of whose members cover X, or equivalently, if every regular closed cover of X has a finite subcover. Herrmann [3] proved that a Hausdorff space is S-closed if and only if it is quasi-H-closed and extremally disconnected. Recall that a space X is said to be quasi-H-closed if every open cover of X has a finite subfamily the closures of whose members cover X. If we replace in the definition of quasi-H-closedness "every open cover" by "every countable open cover" we obtain the important class of feebly compact spaces (also known as lightly compact spaces).

In this paper we introduce and study a new class of spaces, namely countably S-closed spaces, i. e. spaces in which every countable regular closed cover has a finite subcover. In Section 2 we provide several characterizations of countably S-closed spaces and investigate their basic properties. It is pointed out that this class of spaces lies strictly between the

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class of S-closed spaces and the class of feebly compact spaces. In Section 3 we further explore the relationship between countably S-closed spaces and feebly compact spaces. In particular, the concept of km-perfect spaces is introduced. Finally, in Section 4 we present several examples to illustrate the results obtained in Section 2 and Section 3.

For a subset A of a topological space (X, τ) we denote the closure of A and the interior of A by clA and intA, respectively. The subspace topology on A is denoted by $\tau|A$. A subset G of (X, τ) is called *regular open* if G = int(clG). $F \subseteq X$ is said to be *regular closed* if $X \setminus F$ is regular open, or equivalently, if F = cl(intF). The families of regular open subsets and regular closed subsets of (X, τ) are denoted by $RO(X, \tau)$ and $RC(X, \tau)$, respectively. $RO(X, \tau)$ is a base for a coarser topology τ_s on X, called the *semi-regularization topology* on X. (X, τ) is said to be *extremally disconnected*, abbreviated e.d., if every regular open set is closed, or equivalently, if $RO(X, \tau) = RC(X, \tau)$. It is known that every dense subspace of an e.d. space is e.d.. In order to facilitate an easy reading of this paper we now summarize some well known results.

Lemma 1.1 Let (X, τ) be a space. Then

i) $RC(X, \tau) = RC(X, \tau_s)$, ii) (X, τ) is e.d. if and only if (X, τ_s) is e.d., iii) If $A \subseteq X$ is locally dense, i.e. if $A \subseteq int(clA)$, then $RC(A, \tau|A) = \{F \cap A : F \in RC(X, \tau)\}$.

A subset S of (X, τ) is called *semi-open* [5] (*regular semi-open* [1], respectively) if there is an open set U (a regular open set U, respectively) such that $U \subseteq S \subseteq clU$. A space (X, τ) is called *quasi-H-closed* (*feebly compact, S-closed* [8], respectively) if every open cover (every countable open cover, every semi-open cover, respectively) of (X, τ) has a finite subfamily the closures of whose members cover X. Following Hodel [4], a cellular family in a space (X, τ) is a collection of nonempty, pairwise disjoint open sets. We will denote the set of natural numbers by ω , and $\beta \omega$ is the Stone-Cech compactification of ω . Finally, a sequence $\{A_n : n \in \omega\}$ of subsets of a set X is called decreasing (increasing, respectively) if $A_{n+1} \subseteq A_n$ ($A_n \subseteq A_{n+1}$, respectively) for each $n \in \omega$. Strictly decreasing sequences and strictly increasing sequences of subsets are defined in the obvious ways.

No separation axioms are assumed unless explicitly stated.

2 Characterizations and Basic Properties

We begin by defining the class of spaces we will study in this paper.

Definition 1 A topological space (X, τ) is *countably S-closed* if every countable cover of regular closed sets has a finite subcover.

The following fundamental observation is easily verified.

Proposition 2.1 Every S-closed space is countably S-closed, and every countably S-closed space is feebly compact.

Note that the converses of these implications are false, however. Example 4.1 provides a space which is countably S-closed but not S-closed, and in Example 4.3, we present several feebly compact spaces which are not countably S-closed.

In our next result we present a huge variety of characterizations of countably S-closed spaces.

Theorem 2.2 For a space (X, τ) the following are equivalent:

1) (X, τ) is countably S-closed,

2) Every countable cover of X by semi-open sets has a finite subfamily the closures of whose members cover X,

3) Every countable cover of X by regular semi-open sets has a finite subfamily the closures of whose members cover X ,

4) There is no strictly increasing sequence of regular closed sets whose union is X,

5) If $\{F_n : n \in \omega\}$ is a decreasing sequence of nonempty regular closed sets then $\bigcap \{intF_n : n \in \omega\} \neq \emptyset$, 6) If $\{G_n : n \in \omega\}$ is a decreasing sequence of nonempty regular open sets then $\bigcap \{G_n : n \in \omega\} \neq \emptyset$,

7) If $\{G_n : n \in \omega\}$ is a sequence of regular open sets satisfying the finite intersection property then $\bigcap \{G_n : n \in \omega\} \neq \emptyset$,

8) If $\{G_n : n \in \omega\}$ is a filterbasis consisting of regular open sets then $\bigcap \{G_n : n \in \omega\} \neq \emptyset$.

Proof. 1) \Leftrightarrow 2) \Leftrightarrow 3) : This is obvious since the closure of every semi-open set is regular closed. Furthermore, every regular closed set is regular semi-open and thus semi-open.

 $(1) \Rightarrow 4)$: This is trivial.

4) \Rightarrow 1) : Suppose that (X, τ) is not countably S-closed. Then there exists a countable regular closed cover $\{F_n : n \in \omega\}$ of X such that for all $k \in \omega$, $\bigcup \{F_n : 1 \le n \le k\} \ne X$. By induction we can construct a family $\{A_n : n \in \omega\}$ as follows : for n = 1 set $A_1 = F_1$. For $n \ge 2$ there must be a least $m \in \omega$ such that A_{n-1} is strictly contained in $F_1 \cup ... \cup F_m \ne X$. Define A_n by $A_n = F_1 \cup ... \cup F_m$. Since $\{A_n : n \in \omega\}$ is a strictly increasing sequence of regular closed sets whose union is X, we have a contradiction to 4).

 $1) \Rightarrow 5)$: Let $\{F_n : n \in \omega\}$ be a decreasing sequence of nonempty regular closed sets. Suppose that $\bigcap \{intF_n : n \in \omega\} = \emptyset$. Then $\{cl(X \setminus F_n) : n \in \omega\}$ is a regular closed cover of X. By assumption, there exists $m \in \omega$ such that $X = \bigcup \{cl(X \setminus F_i) : i = 1, ..., m\} = cl(X \setminus F_m)$. Hence $intF_m = \emptyset$, which gives a contradiction.

 $(5) \Rightarrow 6)$: Set $F_n = clG_n$ for all $n \in \omega$ and apply (5).

6) \Rightarrow 7) : Let $\{G_n : n \in \omega\}$ be a sequence of regular open sets satisfying the finite intersection property. Set $U_n = G_1 \cap ... \cap G_n$ for all $n \in \omega$. Then $\{U_n : n \in \omega\}$ is a decreasing sequence of nonempty regular open sets and $\bigcap \{G_n : n \in \omega\} = \bigcap \{U_n : n \in \omega\} \neq \emptyset$.

 $(7) \Rightarrow 8$): This is trivial since every filterbase satisfies the finite intersection property.

8) \Rightarrow 1) : Suppose that (X, τ) is not countably S-closed. Then there is a countable regular closed cover $\{F_n : n \in \omega\}$ of X without a finite subcover. For $n \in \omega$ define G_n by $G_n = X \setminus (F_1 \cup ... \cup F_n)$. Then G_n is nonempty and regular open for all $n \in \omega$. Furthermore, it is easily proved that $\{G_n : n \in \omega\}$ is a filterbase with empty intersection, a contradiction to 8). \Box

As an immediate consequence of Lemma 1.1 we note the following result.

Lemma 2.3 Let (X, τ) be a space and suppose that $X = A_1 \cup ... \cup A_n \cup E$, where each A_i is a locally dense, countably S-closed subspace and $E \subseteq X$ is finite. Then (X, τ) is countably S-closed.

Lemma 2.4 Let (X, τ) be a space. Suppose there exists $x_0 \in X$ having an open neighbourhood base $\{U_n : n \in \omega\}$ with the following properties:

- i) each clU_{n+1} is strictly contained in U_n ,
- ii) $U_1 = X$,
- iii) $\{x_0\} = \bigcap \{U_n : n \in \omega\} = \bigcap \{c | U_n : n \in \omega\}$.
- Then (X, τ) is not countably S-closed.

Proof. Let $\{\omega_k : k \in \omega\}$ be a partition of ω where each ω_k is infinite. For every $k \in \omega$ let $G_k = \bigcup \{U_n \setminus clU_{n+1} : n \in \omega_k\}$. Then $\{G_k : k \in \omega\}$ is a cellular family. One checks easily that $x_0 \in clG_k$ and $\bigcup \{clU_n \setminus clU_{n+1} : n \in \omega_k\} \subseteq clG_k$ for each $k \in \omega$. We now show that $\{clG_k : k \in \omega\}$ covers X. If $x \neq x_0$ then there exists $m \in \omega$ such that $x \in clU_m \setminus clU_{m+1}$. There is some $k \in \omega$ such that $m \in \omega_k$ and so $x \in clG_k$. Since $\{G_k : k \in \omega\}$ is a cellular family, $\{clG_k : k \in \omega\}$ is a countable regular closed cover of X without a finite subcover. Thus (X, τ) is not countably S-closed. \Box

Corollary 2.5 1) An infinite regular space which is first countable at some non-isolated point is not countably S-closed.

2) Suppose that (X, τ) is an infinite, regular feebly compact space and there exists $x_0 \in X$ such that $\{x_0\}$ is a G_{δ} -set but not open. Then (X, τ) is not countably S-closed.

Proof. 1) is an immediate consequence of Lemma 2.4. To prove 2) observe that by Proposition 2.2. in [6], (X, τ) is first countable at x_0 . Now apply 1). \Box

We now focus on the fundamental properties of countably S-closed spaces. To begin with, recall that a topological property R is said to be *semi-regular* provided that a space (X, τ) has property R if and only if (X, τ_s) has property R. The property R is called *contagious* if a space (X, τ) has property R whenever a dense subspace of (X, τ) has property R. Our first result is an immediate consequence of Lemma 1.1. **Proposition 2.6** Let R be the property "countably S-closed". Then R is both semi-regular and contagious.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is called *irresolute* if $f^{-1}(S)$ is semi-open in (X, τ) whenever S is semi-open in (Y, σ) . It is known that a function which is continuous, open and onto, is irresolute. Thompson [9] has shown that if (X, τ) is S-closed and $f : (X, \tau) \to (Y, \sigma)$ is irresolute and onto, then (Y, σ) is S-closed. The same idea works to prove our next result.

Proposition 2.7 i) Let (X, τ) be countably S-closed and let $f : (X, \tau) \to (Y, \sigma)$ be irresolute and onto. Then (Y, σ) is countably S-closed.

ii) Let (X, τ) be countably S-closed and let $f : (X, \tau) \to (Y, \sigma)$ be continuous, open and onto. Then (Y, σ) is countably S-closed.

iii) If a product of topological spaces is countably S-closed, then each factor space is countably S-closed.

Remark 2.8 The converse of Proposition 2.7 iii) is false. $\beta\omega$ is S-closed hence countably S-closed, but $\beta\omega \times \beta\omega$ is not countably S-closed as shown in Example 4.4.

Proposition 2.9 Let (X, τ) be countably S-closed.

i) If $G \in RO(X, \tau)$, then $(G, \tau | G)$ is countably S-closed.

ii) If $F \in RC(X, \tau)$, then $(F, \tau | F)$ is countably S-closed.

iii) If $(A, \tau | A)$ is a countably S-closed subspace of (X, τ) (here (X, τ) need not be countably S-closed), and if $A \subseteq T \subseteq clA$, then $(T, \tau | T)$ is countably S-closed.

iv) Let (X, τ) be regular. If $p \in X$ is a non-isolated point, then $X \setminus \{p\}$ is a countably S-closed subspace.

Proof.

i) Let $\{A_n : n \in \omega\} \subseteq RC(G, \tau | G)$ be a cover of G. By Lemma 1.1, for each $n \in \omega$ $A_n = G \cap F_n$ for some $F_n \in RC(X, \tau)$. Since $\{F_n : n \in \omega\} \cup \{X \setminus G\}$ is a regular closed cover of (X, τ) , there exists $m \in \omega$ such that $X = (X \setminus G) \cup F_1 \cup \ldots \cup F_m$. Consequently, $G = A_1 \cup \ldots \cup A_m$ and thus $(G, \tau | G)$ is countably S-closed.

ii) Let $F \in RC(X, \tau)$. Then $intF \in RO(X, \tau)$ and intF is dense in $(F, \tau|F)$. By i) and Proposition 2.6, $(T, \tau|T)$ is countably S-closed.

iii) Since A is dense in $(T, \tau | T)$, by Proposition 2.6, $(T, \tau | T)$ is countably S-closed.

iv) Let $D = X \setminus \{p\}$. If D is finite, clearly $(D, \tau | D)$ is countably S-closed. Suppose that D is infinite. Let $\{A_n : n \in \omega\} \subseteq RC(D, \tau | D)$ be a cover of D. By Lemma 1.1, , for each $n \in \omega$ $A_n = D \cap F_n$ for some $F_n \in RC(X, \tau)$. If $X \neq \bigcup \{F_n : n \in \omega\}$, then $\{p\}$ is a G_{δ} -set in (X, τ) and by Corollary 2.5, (X, τ) and $(D, \tau | D)$ are finite spaces, which is a contradiction. Thus $X = \bigcup \{F_n : n \in \omega\}$, and there exists $m \in \omega$ such that $X = F_1 \cup \ldots \cup F_m$ and $D = A_1 \cup \ldots \cup A_m$, i.e. $(D, \tau | D)$ is countably S-closed. \Box

Remark 2.10 The property "countably S-closed" is in general not hereditary with respect to open, dense or closed subspaces. $\beta\omega$ is S-closed hence countably S-closed. $\omega \subseteq \beta\omega$ is open and dense in $\beta\omega$ but clearly not countably S-closed. Moreover, we show in Example 4.5 that $\beta\omega \setminus \omega$ fails to be countably S-closed.

3 Countably S-closed spaces vs. feebly compact spaces

In this section we focus on the relationship between countably S-closed spaces and feebly compact spaces. We already pointed out in Proposition 2.1 that every countably S-closed space is feebly compact whereas the converse does not hold in general (see Example 4.3). Therefore it is quite natural to search for a condition (P) such that a space is countably S-closed if and only if it is feebly compact and satisfies (P). For the class of S-closed spaces there exists the following interesting result [3] : A Hausdorff space is S-closed if and only if it is quasi-H-closed and e.d. Unfortunately, there is no analogous result for the class of countably S-closed spaces. It is obvious that every feebly compact e.d. space is countably S-closed but in Example 4.2 we show that there exist countably S-closed, compact Hausdorff spaces which are not e.d. . We are, however, able to characterize the class of spaces which are countably S-closed and e.d. .

Definition 2 A space (X, τ) is called *km-perfect* if for each $U \in RO(X, \tau)$ and each $x \notin U$ there is a sequence $\{G_n : n \in \omega\}$ of open sets such that $\bigcup \{G_n : n \in \omega\} \subseteq U \subseteq \bigcup \{clG_n : n \in \omega\}$ and $x \notin \bigcup \{clG_n : n \in \omega\}$.

Our next result shows that there is a variety of spaces which are km-perfect. Recall that a space (X, τ) is said to be *perfect* (*RC-perfect* [6], respectively) if every open set is the countable union of closed sets (regular closed sets, respectively).

Theorem 3.1 If a space (X, τ) is either

i) e.d., or

- ii) hereditarily Lindelöf and Hausdorff, or
- iii) second countable and Hausdorff, or
- iv) *RC*-perfect, or
- v) regular and perfect,

then it is km-perfect.

Proof. Let $U \in RO(X, \tau)$ and $x \notin U$.

i) Suppose that (X,τ) is e.d. . Then U is closed and we are done by setting $G_n=U$ for each $n\in\omega$.

ii) If (X, τ) is hereditarily Lindelöf and Hausdorff, for each $y \in U$ there is an open set V_y such that $y \in V_y \subseteq U$ and $x \notin clV_y$. Then $\{V_y : y \in U\}$ is an open cover of U which possesses a countable subcover $\{V_{y_n} : n \in \omega\}$. Then $U = \bigcup\{V_{y_n} : n \in \omega\}$ and $x \notin clV_{y_n}$ for each $n \in \omega$, proving that (X, τ) is km-perfect.

iii) This follows from ii) since every second countable space is hereditarily Lindelöf.

iv) If (X, τ) is *RC*-perfect then $U = \bigcup \{F_n \in RC(X, \tau) : n \in \omega\}$. Thus $\bigcup \{intF_n \in RC(X, \tau) : n \in \omega\} \subseteq U \subseteq \bigcup \{F_n \in RC(X, \tau) : n \in \omega\}$ and $x \notin F_n$ for each $n \in \omega$.

v) Suppose that (X, τ) is regular and perfect. Then $U = \bigcup \{A_n : n \in \omega\}$ where each A_n is closed. For each $n \in \omega$, $x \notin A_n$ and by regularity there exists an open set G_n with $A_n \subseteq G_n \subseteq U$ and $x \notin clG_n$. Hence (X, τ) is km-perfect. \Box

The importance of the class of km-perfect spaces is illustrated by

Theorem 3.2 Let (X, τ) be countably S-closed and km-perfect. Then (X, τ) is e.d..

Proof. Let $U \in RO(X,\tau)$ and $x \notin U$. Let $\{G_n : n \in \omega\}$ be a sequence of open sets with $\bigcup \{G_n : n \in \omega\} \subseteq U \subseteq \bigcup \{clG_n : n \in \omega\}$ and $x \notin \bigcup \{clG_n : n \in \omega\}$. By Lemma 1.1, $\{U \cap clG_n : n \in \omega\} \subseteq RC(U,\tau|U)$ is a cover of U. By Proposition 2.9, $(U,\tau|U)$ is countably S-closed so there exists $m \in \omega$ such that $U \subseteq clG_1 \cup \ldots \cup clG_m$. Since $x \in X \setminus (clG_1 \cup \ldots \cup clG_m)$, we have $x \notin clU$. Thus U is closed, i.e. (X,τ) is e.d. . \Box

Corollary 3.3 i) A km-perfect space is countably S-closed if and only if it is feebly compact and e. d. .

ii) A countably S-closed space is e. d. if and only if it is km-perfect.

In order to characterize countably S-closed spaces in terms of feebly compact spaces satisfying an additional condition, we need

Lemma 3.4 For a space (X, τ) the following are equivalent:

1) (X, τ) is feebly compact.

2) Every locally finite cellular family is finite.

3) If $\{U_n : n \in \omega\}$ is a decreasing sequence of nonempty open sets (regular open sets, respectively), then $\bigcap \{clU_n : n \in \omega\} \neq \emptyset$.

4) If $\{F_n : n \in \omega\}$ is a decreasing sequence of nonempty regular closed sets, then $\bigcap\{F_n : n \in \omega\} \neq \emptyset$.

Proof. 1) \Leftrightarrow 2) \Leftrightarrow 3) can be found in [7], page 50, and 3) \Leftrightarrow 4) is obvious. \Box

Using Theorem 2.2 and Lemma 3.4 the next result is immediate.

Theorem 3.5 A space (X, τ) is countably S-closed if and only if it is feebly compact and whenever $\{F_n : n \in \omega\}$ is a decreasing sequence of nonempty regular closed sets with nonempty intersection then $\bigcap \{intF_n : n \in \omega\} \neq \emptyset$. **Theorem 3.6** For a space (X, τ) the following are equivalent :

1) (X, τ) is countably S-closed.

2) Every cellular family $\{U_{\lambda} : \lambda \in \Lambda\}$ satisfying $cl(\bigcup\{U_{\lambda} : \lambda \in \Lambda\}) = \bigcup\{clU_{\lambda} : \lambda \in \Lambda\}$ is finite.

Proof.

1) \Rightarrow 2) : Let $\{G_{\lambda} : \lambda \in \Lambda\}$ be a cellular family with $cl(\bigcup\{G_{\lambda} : \lambda \in \Lambda\}) = \bigcup\{clG_{\lambda} : \lambda \in \Lambda\}$. Suppose that Λ is infinite. Pick a countably infinite subset $\Lambda_1 \subseteq \Lambda$ and let $\Lambda_2 = \Lambda \setminus \Lambda_1$. Set $U_{\lambda} = G_{\lambda}$ for each $\lambda \in \Lambda_1$, $U^* = \bigcup\{G_{\lambda} : \lambda \in \Lambda_2\}$ and $V = int(cl(\bigcup\{G_{\lambda} : \lambda \in \Lambda\}))$. Because of $cl(\bigcup\{G_{\lambda} : \lambda \in \Lambda\}) = \bigcup\{clG_{\lambda} : \lambda \in \Lambda\}$ we have $V \subseteq \bigcup\{U_{\lambda} : \lambda \in \Lambda_1\} \cup clU^*$. Since V is a regular open subset it follows from Proposition 2.9 that $(V, \tau | V)$ is countably S-closed. By Lemma 1.1, $\{clU_{\lambda} \cap V : \lambda \in \Lambda_1\} \cup \{clU^* \cap V\} \subseteq RC(V, \tau | V)$, hence there is a finite subset $\{\lambda_1, ..., \lambda_m\} \subseteq \Lambda_1$ such that $V \subseteq clU_{\lambda_1} \cup ... \cup clU_{\lambda_m} \cup clU^*$. If $\lambda \in \Lambda_1 \setminus \{\lambda_1, ..., \lambda_m\}$, then $G_{\lambda} \subseteq V$ and $G_{\lambda} \cap (U_{\lambda_1} \cup ... \cup U_{\lambda_m} \cup U^*)$ is empty, thus G_{λ} is empty, a contradiction. Hence Λ has to be finite.

 $2) \Rightarrow 1)$: If (X, τ) is not countably S-closed then by 4) in Theorem 2.2 there is a strictly increasing sequence $\{F_n : n \in \omega\}$ of regular closed sets whose union is X. Define $U_1 = intF_1$ and $U_n = intF_n \setminus F_{n-1}$ for each $n \ge 2$. It is easily checked that $\{U_n : n \in \omega\}$ is an infinite cellular family satisfying $cl(\bigcup\{U_n : n \in \omega\}) = \bigcup\{clU_n : n \in \omega\}$, a contradiction. Hence (X, τ) is countably S-closed. \Box

Note that condition 2) of the above theorem is a generalization of the condition 2) of Lemma 3.4 because every locally finite family is closure-preserving. Thus it might be interesting to know whether the condition "Every closure-preserving cellular family is finite" defines a new class of spaces between the countably S-closed spaces and the feebly compact spaces.

Recall that a space (X, τ) is called a *P*-space if every G_{δ} -set is open. Note that if X is an uncountable set endowed with the co-countable topology τ , then (X, τ) is a countably *S*-closed *P*-space. There is an interesting characterization of countably *S*-closed *P*-spaces which seems to be worth mentioning. **Proposition 3.7** A *P*-space (X, τ) is countably *S*-closed if and only if every dense subspace is feebly compact.

Proof. Let (X, τ) be countably S-closed and let $D \subseteq X$ be dense. If $\{U_n : n \in \omega\} \subseteq \tau$ is a cover of D then $\bigcup \{clU_n : n \in \omega\}$ is closed and thus equal to X. Hence there exists $m \in \omega$ such that $X = clU_1 \cup ... \cup clU_m$. Consequently, $(D, \tau | D)$ is feebly compact. To prove the converse let $\{F_n : n \in \omega\}$ be a regular closed cover of (X, τ) . Then $\bigcup \{intF_n : n \in \omega\}$ is dense and, by assumption, feebly compact. This clearly implies that (X, τ) is covered by finitely many F_n , i. e. (X, τ) is countably S-closed. \Box

Remark 3.8 Closing this section, we quickly discuss countably S-closed spaces in relationship to first countability and second countability. It is well known that every first countable, e. d. Hausdorff space is discrete (see [11], page 301), and thus every first countable, e. d., countably S-closed Hausdorff space has to be finite. Moreover, since every second countable Hausdorff space is km-perfect it follows by Theorem 3.2 and the preceding observation that every second countable, countably S-closed Hausdorff space is finite. This result is false, however, in the absence of Hausdorffness since the space obtained by taking the cofinite topology on a countably infinite set is obviously non-Hausdorff, second countable and countably S-closed.

4 Examples

Example 4.1 Let $X = \beta \omega \setminus \{p\}$ where $p \in \beta \omega \setminus \omega$. It is well known (see e.g. [11], page 301) that X is countably compact, and hence feebly compact, but not compact. Since X is e.d., X is countably S-closed. However, X fails to be S-closed since a regular S-closed space is compact.

Example 4.2 Let (Y, σ) be a space such that $Y \setminus \{p\}$ is a countably S-closed subspace for some non-isolated point $p \in Y$. Let Y_1 and Y_2 denote two disjoint copies of $Y \setminus \{p\}$. For any subset $A \subseteq Y$ we will denote the corresponding subsets of Y_1 and Y_2 by A_1 and A_2 , respectively. Now let $X = Y_1 \cup Y_2 \cup \{p\}$. We define a topology τ on X in the following way. For any $x \in X$, if $x \in Y_1$ ($x \in Y_2$, respectively), then the basic open neighbourhoods of xin (X, τ) are of the form V_1 (V_2 , respectively) where V is an open subset of $Y \setminus \{p\}$. For every open neighbourhood W of p in (Y, σ) , a basic open neighbourhood of p in (X, τ) is $\{p\} \cup (W \setminus \{p\})_1 \cup (W \setminus \{p\})_2$. It is easy to see that both Y_1 and Y_2 are regular open subsets of (X, τ) and homeomorphic to $Y \setminus \{p\}$. By Lemma 2.3, (X, τ) is countably S-closed but not e.d. since neither Y_1 nor Y_2 are closed in (X, τ) .

In particular, if (Y, σ) is $\beta \omega$ then the resulting space is a compact, countably S-closed Hausdorff space which is not S-closed since it fails to be e.d. .

Example 4.3 Here we present some familiar spaces which are feebly compact but not countably *S*-closed.

i) Is bell's space Ψ [2], page 79, is a locally compact, feebly compact, perfect Hausdorff space hence also completely regular. It is also first countable and thus cannot be countably *S*-closed by Corollary 2.5.

ii) ω_1 , the space of all countable ordinals with the order topology is regular, first countable and countably compact, thus feebly compact. By Corollary 2.5, ω_1 is not countably S-closed.

iii) Let D be an infinite set with the discrete topology. Let (X, τ) denote the one-pointcompactification of D, where $X = D \cup \{a\}$ and $a \notin D$ is the only non-isolated point of (X, τ) . . Then (X, τ) is a compact Hausdorff space, hence feebly compact. Let $\{D_n : n \in \omega\}$ be a partition of D where each D_n is infinite. For each $n \in \omega$, if $F_n = D_n \cup \{a\}$, then $F_n \in RC(X, \tau)$. Clearly, $\{F_n : n \in \omega\}$ is a regular closed cover of (X, τ) without a finite subcover. Thus (X, τ) is not countably S-closed.

Example 4.4 $\beta \omega \times \beta \omega$ is not countably S-closed.

Consider $W = \{(n, n) : n \in \omega\}$. It is very well known that W is a regular open subset of $\beta \omega \times \beta \omega$. By Proposition 2.9 i), $\beta \omega \times \beta \omega$ cannot be countably S-closed since W is also an infinite discrete subspace of $\beta \omega \times \beta \omega$.

Example 4.5 $\omega^* = \beta \omega \setminus \omega$ is not countably S-closed.

Let $f : \omega \to [0,1]$ be a function which maps ω onto the rationals of the unit interval [0,1]. If $\beta f : \beta \omega \to [0,1]$ denotes the Stone-extension of f, then βf is continuous and

onto. Let $g: \omega^* \to [0,1]$ denote the restriction of βf to ω^* , i.e. $g = \beta f | \omega^*$. Note that for each irrational number $t \in [0,1]$ we have $g^{-1}(\{t\}) \neq \emptyset$. By Corollary 2.5, [0,1] is not countably S-closed so there exists a regular closed cover $\{A_n : n \in \omega\}$ of [0,1] without a finite subcover. Clearly each A_n is a zero-set in [0,1], and so each $g^{-1}(A_n)$ is a nonempty zero-set in ω^* . By [10], page 78, $\{g^{-1}(A_n) : n \in \omega\}$ is a countable regular closed cover of ω^* . Let $m \in \omega$. Then there is an irrational number $t \in [0,1] \setminus (A_1 \cup ... \cup A_m)$. Since $g^{-1}(\{t\}) \neq \emptyset$, we have $\omega^* \neq g^{-1}(A_1) \cup ... \cup g^{-1}(A_m)$. This proves that ω^* is not countably S-closed.

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