A NOTE ON EXTENSIONS GENERATED BY CLOSED SETS

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appeared in: Tamkang J. Math. 26 (2) (1995), 125–129.

Abstract

In a recent paper Abd El-Monsef et al. consider a certain topology on 2^X where 2^X is the family of all nonempty closed subsets of a given topological space X . Unfortunately, several results in their paper are incorrect and so the purpose of this note is to correct, improve and expand these results. In addition, the main question in their paper turns out to have a quite simple answer.

1 Introduction and Preliminaries

For a topological space (X, τ) the closure of a subset A of X is denoted by $cl_\tau A$ and we will suppress the τ when there is no confusion possible. A space (X, τ) is called R_0 if for any open set U containing a point x we have $cl\{x\} \subseteq U$. A subset S of (X, τ) is called *locally closed* [2] if S is the intersection of an open set and a closed set, or equivalently, if $S = U \cap clS$ for some open set U . We will call a subset S of (X, τ) an F_α*-set* if S is the union of closed subsets of (X, τ) .

Observation 1.1 A space (X, τ) is T_1 (resp. R_0) if and only if every subset (resp. every open subset) is an F_α-set.

No separation axioms are assumed unless stated explicitly. Finally, the set of natural numbers is denoted by ω .

2 The correspondence $U \to U^*$

Following Michael [3], for a space (X,τ) let 2^X be the family of all nonempty closed subsets of (X,τ) .

Definition 1 For each open subset U of a space (X,τ) let $U^* = \{F \in 2^X \ : \ F \subseteq U\}$.

We now investigate the correspondence $U \to U^*$ for arbitrary spaces (X,τ) . It turns out that ii), iii) and iv) of Theorem 2.1 in [1] are incorrect and have to be modified.

Observation 2.1 If U and V are open subsets of (X,τ), then

1) $\emptyset^* = \emptyset$ and $X^* = 2^X$,
2) $(U \cap V)^* = U^* \cap V^*$,
3) if $U \subseteq V$ then $U^* \subseteq V^*$,
4) $U^* \cup V^* \subseteq (U \cup V)^*$.

Consider the following conditions where U and V are arbitrary open sets in (X,τ) :

$(P1)$ $U^* \subseteq V^*$ implies $U \subseteq V$,
$(P2)$ $U \neq \emptyset$ implies $U^* \neq \emptyset$,
$(P3)$ $(U \cup V)^* \subseteq U^* \cup V^*$.

It turns out that these conditions have quite useful characterizations. First observe that for open subsets U and V of (X,τ) , $U \not\subseteq V$ if and only if $U \setminus V$ is nonempty and locally closed. In addition, $U^* \not\subseteq V^*$ if and only if there is a nonempty closed set contained in $U \setminus V$.

Theorem 2.2 For a space (X,τ) the following are equivalent :

1) (X,τ) satisfies $(P1)$,
2) every nonempty locally closed set A contains an F_α-set which is dense in A .

Proof.

1) $\Rightarrow$ 2) : Let $A \subseteq X$ be locally closed, i.e. $A = U \cap clA$ for some open set U . If $S = \bigcup\{F \subseteq X \ : \ F$ is closed and $F \subseteq A\}$ then S is an F_α-set and $S \subseteq A$. We claim that $A \subseteq clS$. Suppose there exists $x \in A$ with $x \notin clS$. Then there exists an open set G containing x with $G \subseteq U$ and $G \cap S = \emptyset$. Now pick $F \in G^*$. If $F \cap clA$ is nonempty then $F \cap clA \subseteq U \cap clA = A$ and so $F \cap clA \subseteq S$, a contradiction. Hence $F \cap clA = \emptyset$ and so $G^* \subseteq (X \setminus clA)^*$. By condition $(P1)$ we have $G \subseteq X \setminus clA$ and $G \cap A = \emptyset$, a contradiction. Thus $A \subseteq clS$ and we are done.

2) $\Rightarrow$ 1) : Let $U, V \subseteq X$ be open with $U^* \subseteq V^*$. Suppose that $U \nsubseteq V$. If $A = U \cap (X \setminus V)$ then A is nonempty and locally closed, so there exists an F_α-set S with $S \subseteq A \subseteq clS$. In particular there exists $F \in 2^X$ with $F \subseteq S$. Clearly $F \in U^*$ and $F \notin V^*$, a contradiction. Thus $U \subseteq V$. $\square$

Theorem 2.3 For a space (X, τ) the following are equivalent :

1) (X, τ) satisfies $(P2)$,

2) every nonempty open set U contains an F_α-set S which is dense in U .

Proof. 1) $\Rightarrow$ 2) : Let $U \subseteq X$ be open and let $S = \bigcup\{F \subseteq X \ : \ F$ is closed and $F \subseteq U\}$. Then S is an F_α-set with $S \subseteq U$. We claim that $U \subseteq clS$. Suppose that $U \cap (X \setminus clS)$ is nonempty. By condition $(P2)$ there exists $F \in 2^X$ with $F \subseteq U$ and $F \cap S = \emptyset$, a contradiction. Thus $U \subseteq clS$ and we are done.

2) $\Rightarrow$ 1) : Let $U \neq \emptyset$ be open. By assumption there exists an F_α-set S with $S \subseteq U \subseteq clS$. In particular there exists $F \in 2^X$ with $F \subseteq S$ and so $F \in U^*$, i.e. $U^* \neq \emptyset$. $\square$

Theorem 2.4 For a space (X, τ) the following are equivalent :

1) (X, τ) satisfies $(P3)$,

2) if $F_1, F_2 \in 2^X$ then $F_1 \cap F_2 \neq \emptyset$,

3) if $U \neq X$ is open then $U^* = \emptyset$.

Proof.

1) $\Rightarrow$ 2) : Let $F_1, F_2 \in 2^X$ and suppose that $F_1 \cap F_2 = \emptyset$. If $U = X \setminus F_1$, $V = X \setminus F_2$ and $F = F_1 \cup F_2$ then U and V are open with $U \cup V = X$ and so $F \in (U \cup V)^*$, i.e. $F \subseteq X \setminus F_1$ or $F \subseteq X \setminus F_2$, a contradiction. Hence $F_1 \cap F_2 \neq \emptyset$.

2) $\Rightarrow$ 3) : Let $U \neq X$ be open and suppose there exists $F \in U^*$. Since $X \setminus U$ is nonempty and closed, we have $F \cap (X \setminus U) = \emptyset$, a contradiction. Thus $U^* = \emptyset$.

3) $\Rightarrow$ 2) : Let $F_1, F_2 \in 2^X$. By assumption $(X \setminus F_1)^* = \emptyset$ and so $F_2 \notin (X \setminus F_1)^*$, i.e. $F_1 \cap F_2 \neq \emptyset$.

2) $\Rightarrow$ 1) : Let $U, V \subseteq X$ be open and let $F \in (U \cup V)^*$. Suppose that $F \notin U^* \cup V^*$. If $F_1 = F \cap (X \setminus U)$ and $F_2 = F \cap (X \setminus V)$ then $F_1, F_2 \in 2^X$. By assumption $F_1 \cap F_2 \neq \emptyset$, a contradiction. Thus $F \in U^* \cup V^*$. $\Box$

Corollary 2.5 If (X, τ) is T_1 and $|X| \geq 2$, then (X, τ) does not satisfy $(P3)$.

From the previous results it is clear that for a space (X, τ) the following implications hold :

$$T_1 \Rightarrow R_0 \Rightarrow (P1) \Rightarrow (P2).$$

We will now point out that none of these implications is reversible and that there are spaces which do not satisfy $(P2)$. First note that the indiscrete topology on an infinite set yields an R_0 space which is not T_1 .

Example 2.6 Let X be an infinite set and let $p \in X$. It is clear that $\tau = \{\emptyset\} \cup \{G \subseteq X :$ $p \in G$ and $X \setminus G$ is finite $\}$ is a topology on X . Note that $\{x\}$ is closed whenever $x \neq p$ and that $\{p\}$ is dense. Hence (X, τ) is not R_0 . We now show that (X, τ) satisfies $(P1)$. Let $U, V \subseteq X$ be open with $U^* \subseteq V^*$ and let $x \in U$. If $x \neq p$ then $\{x\} \in U^*$ and so $x \in V$. If $x = p$ then there exists $y \neq p$ with $y \in U$, and so $\{y\} \in U^*$, hence $V \neq \emptyset$ and $p \in V$. Thus $U \subseteq V$.

Example 2.7 Let $\mathbb{R}$ denote the set of reals and let $X = \mathbb{R} \cup \{p\}$ where $p \notin \mathbb{R}$. A topology τ on X is defined as follows. Basic neighbourhoods of $x \in \mathbb{R}$ are of the form $(x - \varepsilon, x + \varepsilon)$ where $\varepsilon > 0$. A basic neighbourhood of p is of the form $\{p\} \cup (x - \varepsilon, x + \varepsilon)$ with $\varepsilon > 0$. It is easy to check that (X, τ) satisfies $(P2)$ (in fact, every open set contains a suitable τ-closed interval $[a, b]$) . If $F \subseteq \mathbb{R}$ is nonempty and closed, then $p \notin F$ and thus $0 \notin F$. So $\mathbb{R}^* \subseteq (\mathbb{R} \setminus \{0\})^*$ but obviously $\mathbb{R}$ is not contained in $\mathbb{R} \setminus \{0\}$, i.e. (X, τ) does not satisfy $(P1)$.

Example 2.8 Let τ be the following topology on ω , $\tau = \{\emptyset\} \cup \{ \{1, ..., n\} \ : \ n \in \omega\}$. By Theorem 2.4 it is clear that (ω, τ) satisfies $(P3)$. Obviously (ω, τ) does not satisfy $(P2)$.

3 The Space $(2^X, \sigma)$

It is well known that, given a space (X, τ), one may define several quite interesting topologies on 2^X , see e.g. [3] . In [4], Schmidt discusses a certain topology σ on (X, τ) which has already been mentioned by Michael in [3]. In their recent paper Abd El-Monsef et al. [1] continued the study of this space $(2^X, \sigma)$ and we will now improve and correct some of their results.

Given a space (X, τ), it is clear from Observation 2.1 that $\{U^* \ : \ U \subseteq X$ is open in $(X, \tau)\}$ is a base for a topology σ on 2^X . This topology σ will be considered throughout this section.

Theorem 3.1 For a space (X, τ) let $F_1, F_2 \in 2^X$. If $F_1 \subseteq F_2$ then $F_2 \in cl_\sigma\{F_1\}$. If (X, τ) is R_0, then the converse is also true.

Proof. Suppose that $F_2 \notin cl_\sigma\{F_1\}$. Then there exists $U \subseteq X$ open in (X, τ) with $F_2 \in U^*$ and $U^* \cap \{F_1\} = \emptyset$, i.e. $F_1 \notin U^*$ and this is a contradiction.

Now let (X, τ) be R_0 and let $F_2 \in cl_\sigma\{F_1\}$. Suppose that there exists $x \in F_1 \setminus F_2$. Then $cl_\tau\{x\} \subseteq X \setminus F_2$ and so $F_2 \in (X \setminus cl_\tau\{x\})^*$. By hypothesis, $F_1 \in (X \setminus cl_\tau\{x\})^*$ and consequently $x \notin F_1$, a contradiction. Hence $F_1 \subseteq F_2$. $\Box$

Corollary 3.2 If (X, τ) is T_1 and $|X| > 1$ then $(2^X, \sigma)$ is not T_1 . Hence Theorem 3.1., Theorem 4.1., Theorem 4.2. and Theorem 4.3. in [1] are false.

Corollary 3.3 Let $W \subseteq 2^X$ be an open set in $(2^X, \sigma)$ containing $X \in 2^X$. Then $W = 2^X$.

Proof. If $F \in 2^X$ then $F \subseteq X$ and so $X \in cl_\sigma\{F\}$, i.e. $F \in W$. $\square$

From Corollary 3.3 we immediately obtain

Corollary 3.4 $(2^X, \sigma)$ is always compact.

Corollary 3.5 $(2^X, \sigma)$ satisfies condition $(P3)$.

Proof.

Let $B_1, B_2 \subseteq 2^X$ be nonempty and closed in $(2^X, \sigma)$. Suppose that $B_1 \cap B_2 = \emptyset$. Then $2^X = (2^X \setminus B_1) \cup (2^X \setminus B_2)$ and we assume w.l.o.g. that $X \in 2^X \setminus B_1$. By Corollary 3.3 we have $2^X \setminus B_1 = 2^X$, i.e. $B_1 = \emptyset$ which is a contradiction. So $B_1 \cap B_2 \neq \emptyset$ and hence $(2^X, \sigma)$ satisfies $(P3)$. $\square$

As a consequence, $(2^X, \sigma)$ is not T_1 whenever $|X| \geq 2$.

In concluding this section we will consider the map $f : (X, \tau) \to (2^X, \sigma)$ where $f(x) = cl_\tau\{x\}$ for each $x \in X$. It has been pointed out in [1] that f is a dense embedding provided that (X, τ) is T_1 . Our final result shows that the converse also holds.

Theorem 3.6 *[**?**]* If $f : (X, \tau) \to (2^X, \sigma)$ where $f(x) = cl_\tau\{x\}$ for each $x \in X$ is an embedding then (X, τ) is T_1 .

Proof. Suppose that $x, y \in X$ with $x \neq y$ and $y \in cl_\tau\{x\}$. Since f is one-to-one, (X, τ) is T_0 and so there exists $U \subseteq X$ open in (X, τ) with $x \in U$ and $y \notin U$. Since $f(U)$ is open in $f(X)$ there exists $V \subseteq X$ open in (X, τ) with $f(x) \in V^* \cap f(X) \subseteq f(U)$. Hence $f(y) \in f(U)$ since $cl_\tau\{y\} \subseteq cl_\tau\{x\}$. As f is one-to-one, we have $y \in U$, a contradiction. This proves that (X, τ) is T_1 . $\square$

Remark 3.7 The pair $((2^X, \sigma), f)$ is a compactification of (X, τ) if and only if (X, τ) is a T_1 space.

References

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