α-Scattered Spaces II*

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Abstract

The aim of this paper is to continue the study of scattered and primarily of α-scattered spaces. The relations between the two concepts and some allied topological properties are investigated. Three problems are left open.

1 Introduction

A topological space X is *scattered* if the only perfect or equivalently crowded subset of X is the empty set. Crowded sets are sometimes called dense-in-themselves and scattered sets are known as dispersed, zerstreut or clairsemé.

For a topological space X, the Cantor-Bendixson derivative $D(X)$ is the set of all non-isolated points of X. It is a fact that the Cantor-Bendixson derivative of every scattered space is nowhere dense. That $D(X)$ is nowhere dense is equivalent to the assumption that $I(X)$, the sets of all isolated points of X, is dense in X. The spaces satisfying this last condition are precisely the spaces whose α-topologies are scattered. These spaces are called in [23] *α-scattered*. The concept was used in [23] to show that in α-scattered spaces the notions of submaximal spaces and α-spaces coincide and thus a recent result of Arhangel'skiĭ and Collins [3] was improved. Topological spaces with countable Cantor-Bendixson derivative were studied in [10]. Such spaces are called d-Lindelöf.

The aim of this paper is to continue the study of scattered and α-scattered spaces.

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2 Scattered and α-scattered spaces

The *perfect kernel* of a given subset A of a topological space X is its largest possible crowded subset. We denote it by $pk(A)$. Clearly $pk(A)$ is determined for every set A, since arbitrary union of crowded sets is crowded. Note that for any subset A of a space X, $pk(A)$ is always a perfect subset of A, and if additionally A is closed in X, then $pk(A)$ is perfect in X. Also, for every set A, $pk(A)$ is a subset of the Cantor-Bendixson derivative $D(A)$.

The *scattered kernel* of a subset A of a space X is the set $sk(A) = A \setminus pk(A)$. Note that $sk(A)$ is always scattered.

It is well-known that every closed set A of a space X has its unique representation as the disjoint union of a perfect and a scattered set. In fact, this representation is as follows: $A = pk(A) \cup sk(A)$.

Lemma 2.1 *Let A be a subset of a space X. Then $I(A) \subseteq sk(A) \subseteq \mathrm{Cl}(I(A))$. The scattered kernel $sk(A)$ is always open in A.*

Proof. Note first that $I(A) = A \setminus D(A) \subseteq A \setminus pk(A) = sk(A)$. Since $A \setminus \mathrm{Cl}(I(A))$ is crowded, then $sk(A) = A \setminus pk(A) \subseteq A \setminus (A \setminus \mathrm{Cl}(I(A))) \subseteq \mathrm{Cl}(I(A))$. Since $pk(A) = A \setminus sk(A)$ is closed in A, then the last claim of the lemma is clear. □

¿From the above given lemma, we have the following characterization of α-scattered spaces.

Theorem 2.2 *For a topological space (X, τ) the following conditions are equivalent:*

(1) *X is α-scattered.*

(2) *The scattered kernel of X is a dense subset of X.* □

Remark 2.3 Even in an α-scattered space, $I(X)$ need not coincide with $sk(X)$. Let $X = \{a, b, c\}$, where the non-trivial open sets are $\{a\}$ and $\{a, b\}$. Clearly $I(X) = \{a\}$ and $sk(X) = X$.

The *α-topology* [21] on a topological space X is usually denoted by τ^α. This is the collection of all subsets of the form $U \setminus N$, where $U \in \tau$ and N is nowhere dense subset of (X, τ).

Theorem 2.4 *If $X = P \cup Q$ denotes the decomposition of a space (X, τ) into its perfect part P and its scattered part Q with respect to the α-topology, then $Q = \mathrm{Int}(\mathrm{Cl}(I(X)))$, where $I(X)$ is the set of all isolated points of (X, τ).*

Proof. By Lemma 2.1, $sk(X) \subseteq \mathrm{Cl}(I(X))$. Thus, $X \setminus \mathrm{Cl}(I(X)) \subseteq pk(X)$. Since every τ-open, τ-crowded set is τ^α-crowded, then $X \setminus \mathrm{Cl}(I(X)) \subseteq P$. Hence, $X \setminus \mathrm{Int}(\mathrm{Cl}(I(X))) = \mathrm{Cl}(X \setminus \mathrm{Cl}(I(X))) = \mathrm{Cl}_\alpha(X \setminus \mathrm{Cl}(I(X))) \subseteq P$ (note that P is τ^α-closed). Thus $Q \subseteq \mathrm{Int}(\mathrm{Cl}(I(X)))$. Now, suppose that $x \in P \cap \mathrm{Int}(\mathrm{Cl}(I(X)))$. Then $I(X) \subseteq Q \cup \{x\} \subseteq \mathrm{Int}(\mathrm{Cl}(I(X)))$, and so $Q \cup \{x\} = V$ is α-open, and $V \cap P = \{x\}$, a contradiction, since P is τ^α-crowded. So, $Q = \mathrm{Int}(\mathrm{Cl}(I(X)))$. □

¿From Theorem 2.4, we immediately have the following result:

Corollary 2.5 [23] *A topological space (X, τ) is α-scattered (i.e. $P = \emptyset$) if and only if $I(X)$ is dense in X.* □

It is not difficult to find examples of α-scattered spaces that are not scattered (see [23, Example 1]). Here is one example from the Euclidean plane.

Example 2.6 Examples of α-scattered spaces that are not scattered can be found in $\mathbf{R}^2$. Let Y be the subset of the plane consisting of all points $(\frac{m}{n}, \frac{1}{n})$, where $n > 0$ and the greatest common divisor of m and n is 1. Clearly Y is discrete. Let $X = \mathrm{Cl}Y$. Note that X is α-scattered, since $I(X) = Y$ but X is not scattered as shown next. Set $A = [0, 1] \times \{0\}$. We will show that $A \subseteq X$. Let x be any rational number from the open interval $(0, 1)$. Then $x = \frac{m}{n}$, where $n > 0$ and 1 is the greatest common divisor of m and n. Clearly for every prime number p there corresponds a number $k \in \{1, \ldots, p-1\}$, such that $|\frac{k}{p} - x| \leq \frac{1}{p}$. Then the distance between $(\frac{k}{p}, \frac{1}{p})$ and $(x, 0)$ is less or equal to $|\frac{k}{p} - x| + \frac{1}{p} \leq \frac{2}{p}$. Hence $(x, 0) \in X$ and consequently $A \subseteq X$. Since A is crowded and nonempty, then X is non-scattered.

The following result settles an open problem from [23].

Theorem 2.7 (i) *Scatteredness is finitely productive.*

(ii) [23] *α-scatteredness is finitely productive.*

(iii) [23] *α-scatteredness is not infinitely productive.*

Proof. Let X and Y be two scattered topological spaces. Assume that $X \times Y$ is not scattered and we will arrive at a contradiction. Let A be a nonempty crowded subset of $X \times Y$ and let p be the canonical projection $X \times Y \to X$. Since $p(A)$ is a nonempty subset of the scattered space X, then there exists a point $a \in p(A)$ and an open set U of X such that $U \cap p(A) = \{a\}$. Then $W(a) = \{y \in Y : (a, y) \in A\}$ is a nonempty subset of the scattered space Y. Thus, there exists a point $b \in W(a)$ and an open set V of Y such that $V \cap W(a) = \{b\}$. Note that $\{(a, b)\} \subseteq (U \times V) \cap A$. Next we show that the reverse inclusion is also true. For let $(x, y) \in (U \times V) \cap A$. Since $(x, y) \in A$, then $x = p(x, y) \in p(A)$ and hence $x \in U \cap p(A) = \{a\}$. Thus $y \in W(x) = W(a)$ and so $y \in V \cap W(a) = \{b\}$. This shows that $(x, y) = (a, b)$; hence $(U \times V) \cap A = \{(a, b)\}$. The last equality shows that A has an isolated point. Contradiction. □

Recall that Arhangel'skiĭ calls a topological property $\mathcal{P}$ *Dutch* [2] if $\mathcal{P}$ is closed hereditary and preserved under finite products. Several classes of spaces have the Dutch property, for example metrizable spaces, developable spaces, first countable spaces. The real line with a topology in which a nonempty proper subset is open iff it consists of rational numbers is an example of an α-scattered space containing a nonempty crowded closed subspace. Thus we have the following result:

Corollary 2.8 *Scatteredness is Dutch property; α-scatteredness is not.* □

Remark 2.9 Henriksen and Isbell call a topological property $\mathcal{P}$ *fitting* (see [4]) if whenever $f\colon (X, \tau) \to (Y, \sigma)$ is a perfect mapping then X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$. Let (X, τ) be the real line with a topology in which the only nontrivial open subsets are the sets of all rational numbers $\mathbf{Q}$ and all irrational numbers $\mathbf{P}$. If (Y, σ) is its subspace $(0, \sqrt{2})$, then the function

$$f(x) = \begin{cases} 0, & x \in \mathbf{Q}, \\ \sqrt{2}, & \text{otherwise.} \end{cases}$$

is perfect, X is crowded (hence not α-scattered) and Y is discrete (hence scattered). Thus neither scatteredness nor α-scatteredness is a fitting property.

It is proved in [23, Theorem 1] that a space X is α-scattered iff every somewhere dense subspace of X has an isolated point iff $I(X)$ is dense in X. The following theorem gives further characterizations of α-scattered spaces.

Theorem 2.10 *For a topological space* (X,τ) *the following conditions are equivalent:*

(1) X *is* α*-scattered.*

(2) $I(X)$ *is dense in the semi-regularization* τ_s*.*

(3) $I(X)$ *is* θ*-dense, i.e.* $\mathrm{Cl}_\theta I(X) = X$*.* $\Box$

A set A has the Baire property (= BP-set) if there is an open set U with $A \triangle U$ (= $(A \setminus U) \cup (U \setminus A)$) meager [17].

Theorem 2.11 *If* X *is* α*-scattered, then every subset of* X *has the Baire property.*

Proof. Let $A \subseteq X$. Note first that $A_1 = A \cap I(X)$ is open in X and hence A_1 has the Baire property. On the other hand $D(X)$ is clearly nowhere dense in X (being closed and codense). Hence its subset $A_2 = A \cap D(X)$ is nowhere dense in X. Clearly A_2 has the Baire property (being meager). Since the BP-sets form σ-algebra on X, then $A = A_1 \cup A_2$ has the Baire property. $\Box$

¿From the above given proof, it is clear that every subset of an α-scattered space is simply-open, since a set is simply-open iff it is union of an open and a nowhere dense set. Observing that a space is strongly irresolvable [9] iff every subset is simply-open, we have the following:

Proposition 2.12 *Every* α*-scattered space is* α*-submaximal, i.e. strongly irresolvable.* $\Box$

As mentioned in [23], the crowded submaximal spaces of Hewitt without isolated points show that there exist strongly irresolvable spaces that fail to be α-scattered.

Not only do all subsets of an α-scattered space have the classical property of Baire, more particularly, they have the property of Baire relative to the ideal $\mathcal{N}(\tau)$ of nowhere dense sets. Let $\mathcal{I}$ be an ideal of subsets of a space (X,τ). A subset A has the property of Baire relative to the ideal $\mathcal{I}$, or more simply A has the $\mathcal{I}$-property of Baire, if A is $\mathcal{I}$-equivalent to a τ-open set, i.e., there is an open set U (in τ) whose symmetric difference with A belongs to $\mathcal{I}$. The family of all subsets having the $\mathcal{I}$-property of Baire is denoted $Br(X, \tau, \mathcal{I})$. It is noted in [26] that for any ideal $\mathcal{I}$, $Br(X, \tau, \mathcal{I})$ is closed under finite intersection and finite union and is a field if and only if $Br(X, \tau, \mathcal{I})$ contains the τ-closed sets. In this case, it is a σ-field if also $\mathcal{I}$ is a σ-ideal. Theorem 1 of the same paper states:

Theorem 2.13 [26] *If $\mathcal{N}(\tau)$ is contained in $\mathcal{I}$, then $Br(X, \tau, \mathcal{I})$ is a field and the converse holds if $\mathcal{I}$ is τ-codense.* $\square$

Recall that the ideal $\mathcal{I}$ is τ-local if $\mathcal{I}$ contains all subsets of X which are locally in $\mathcal{I}$. A subset A is locally in $\mathcal{I}$ if it has an open cover each member of which intersects A in an ideal amount, i.e., each point of A has a neighborhood whose intersection with A belongs to $\mathcal{I}$. This is equivalent to A being disjoint with $A^*(\mathcal{I})$ where $A^*(\mathcal{I}) = \{x \in X$: every neighborhood of x has an intersection with A not belonging to $\mathcal{I}\}$. It is known that $\mathcal{N}(\tau)$ is always τ-local and the classical Banach Category Theorem asserts that the σ-extension of $\mathcal{N}(\tau)$, $\mathcal{M}(\tau)$, the ideal of meager sets is always τ-local. However, $\mathcal{M}(\tau)$ is τ-codense if and only if (X, τ) is a Baire space. It is shown in [24], that every ideal is contained in a smallest local ideal called the local extension. It is shown there that in any T_1-space or perhaps T_D-space, the ideal of scattered sets is the local extension of the ideal of finite sets. It is shown [22] that if an ideal $\mathcal{I}$ is τ-local, then the elements of the topology $\tau^*(\mathcal{I})$ have the simple form $U \setminus E$ where $U \in \tau$ and $E \in \mathcal{I}$. Recall that $\tau^*(\mathcal{I})$ is the smallest extension of τ in which members of $\mathcal{I}$ are closed. When this happens, $\tau^*(\mathcal{I})$ is called simple. Theorem 2 of [26] states:

Theorem 2.14 *If $\tau^*(\mathcal{I})$ is simple and $Br(X, \tau, \mathcal{I})$ is a field, then $Br(X, \tau, \mathcal{I}) = LC(X, \tau^*(\mathcal{I}))$ (the set of locally closed sets for the expansion topology). If also $\mathcal{I}$ is a σ-ideal, $Br(X, \tau, \mathcal{I}) = Bo(X, \tau^*(\mathcal{I}))$ (the set of Borel sets for the expansion topology).* $\square$

Proposition 2.15 *If X is α-scattered, $\mathcal{N}(\tau) = \mathcal{M}(\tau) = \mathcal{P}(D(X))$, where $\mathcal{P}(A)$ is the power set of A.* $\square$

It follows that $Br(X, \tau, \mathcal{N}(\tau))$ is the family of sets which have the classical property of Baire. From the two theorems above we have the following.

Proposition 2.16 *If X is α-scattered, then $\mathcal{P}(X) = LC(X, \tau^\alpha)$, where $\tau^\alpha = \tau^*(\mathcal{N}(\tau))$.* $\Box$

Recall that a family $\mathcal{W}$ of subsets of a topological space (X, τ) is *chain-countable* [16] if every disjoint subfamily of $\mathcal{W}$ is countable. It is proved in [16] that if the family of all open non-meager sets of a topological space (X, τ) is chain-countable, then every point-finite subfamily of the family of all non-meager BP-sets is countable. As a consequence of that result we have the following proposition involving α-scattered spaces.

Proposition 2.17 *If the family of all open 2. category sets of an α-scattered space (X, τ) is chain-countable, then every point-finite family of 2. category sets is countable.* $\Box$

Unlike scatteredness, α-scatteredness is not a hereditary property [23]. However, the following results shows that α-scatteredness is an open hereditary property.

Theorem 2.18 *Every open subspace of an α-scattered space is α-scattered.*

Proof. Let $U \subseteq X$, where U is open (and nonempty) in X and X is α-scattered. Since $I(X)$ is dense in X, then $T = U \cap I(X)$ is nonempty. Clearly $T = I(U)$. If V is a nonempty open subset of U, then V is open in X and hence meets T. Thus T is dense in U and so is $I(U)$ being its superset. Thus U is α-scattered. $\Box$

Corollary 2.19 *Let $\{X_i : i \in I\}$ be a family of spaces and let X denote their topological sum. Then the following conditions are equivalent:*

(1) *X is α-scattered.*

(2) *X_i is α-scattered for each $i \in I$.*

Proof. (1) $\Rightarrow$ (2) follows from the theorem above. (2) $\Rightarrow$ (1) Let U be an open nonempty subset of X and let $U_i = U \cap X_i$. Note that $\cup_{i \in I} I(X_i) \subseteq I(X)$. Clearly for some $k \in I$, U_k is nonempty. Since U_k is open in X_k, then U_k and hence U meets $I(X_k)$. This shows that $U \cap I(X) \neq \emptyset$. Thus $I(X)$ is dense in X and X is α-scattered. $\Box$

Remark 2.20 It is left to the reader to check that Theorem 2.18 can be improved in a way such that the openness of the subspace can be reduced to β-openness. Recall that a subset A of a topological space (X, τ) is *β-open* if A is dense in some regular closed subspace of X.

One consequence by Corollary 2.5 of [6] is that X^α is submaximal if X is α-scattered for certainly then $\mathcal{P}(X)$ contains all τ^α-closed sets.

Theorem 2.21 *A space X is strongly irresolvable if and only if every subset has the property of Baire relative to the ideal of nowhere dense sets.*

Proof. The space X is strongly irresolvable if and only if X^α is submaximal which occurs if and only if $LC(X^\alpha) = Br(X, \tau, \mathcal{N}(\tau)) = \mathcal{P}(X)$. $\Box$

The following corollary answers an open problem of [23] in the positive.

Corollary 2.22 *Strong irresolvability is finitely productive.* $\Box$

Theorem 2.23 *Let $X^\mu = (X, \tau^*(\mathcal{M}(\tau)))$. Then X^μ is submaximal if and only if every subset of X has the (classical) property of Baire.*

Proof. Similar to the above with $LC(X^\mu) = Br(X, \tau)$. $\Box$

Recall that a (nonempty) space (X, τ) is called *hyperconnected* if every nonempty open set is dense.

Proposition 2.24 *Every hyperconnected space X is either crowded or α-scattered.* $\Box$

Often in literature, it is mentioned that the scattered sets of a T_1-space form an ideal (the scattered sets always form a subideal (= h-family [6])). We show next that the separation property T_D (= singletons are locally closed) is enough but first we have the following lemma.

Lemma 2.25 (i) *Every scattered subset of a crowded T_D-space is nowhere dense.*

(ii) *Every dense subset of a crowded T_D-space is also crowded.* $\Box$

Theorem 2.26 *Let X be a T_D-space. Then the collection $\mathcal{S}$ of all scattered subsets of X forms an ideal on X.*

Proof. Heredity is clear even if X is not T_D. Thus we need to verify only finite additivity. For let $S_1,\ S_2 \in \mathcal{S}$ and let A be a crowded subset of $S = S_1 \cup S_2$. We will show that A coincides with the void set. By heredity, $W = A \cap S_1$ is scattered and by Lemma 2.25 (i), W is nowhere dense and hence codense in A. Hence $V = A \setminus S_1$ is dense in A. By Lemma 2.25 (ii), V is crowded. Since V is a subset of the scattered set S_2, then by heredity V is empty. Thus $A \subseteq S_1$. Applying again the heredity of $\mathcal{S}$, we have that $A = \emptyset$. Hence S is scattered and we have the finite additivity of $\mathcal{S}$. □

Here is how α-scatteredness is related to some weak separation properties.

Theorem 2.27 *Every α-scattered space is semi-$T_{\frac{1}{2}}$.*

Proof. In the notion of [27, Theorem 4.8], we need to show that every singleton is semi-open or semi-closed. This is clear, since every point in $I(X)$ is open and every point of $D(X)$ is nowhere dense. □

However, there are α-scattered spaces that fail to be even $T_{\frac{1}{2}}$; even T_1^*. (Every infra-space with an isolated point for instance - but not the Sierpinski space of course.)

Recall that a topological space (X, τ) is called *Katetov T* [18] (for a given separation axiom T) if there is some subtopology of τ that is a minimal T topology on X. A minimal T topology is a topology that satisfies T, but that contains no proper subtopology satisfying T. It is shown in [18] that every scattered space is Katetov T_D. We can easily find an α-scattered space that is not Katetov T_D. The relation between Katetov T and α-scattered spaces is given as follows.

Theorem 2.28 *Every α-scattered space is Katetov semi-T_D.*

Proof. Let (X, τ) be nonempty and α-scattered. Let $x \in I(X)$. Consider the infra topology $\sigma = \{\emptyset, \{x\}, X\}$. Note that σ is a minimal semi-T_D subtopology of τ, since none of the points of the indiscrete topology is open or semi-closed. Thus X is Katetov semi-T_D. □

Local scatteredness is a superfluous concept. It can be easily proved that if every point of a given topological space X has a scattered neighborhood, then X itself is scattered. We can prove the same for α-scattered spaces. (A set is α-scattered if it is α-scattered as a subspace.)

Theorem 2.29 *If every point of a space X has an α-scattered neighborhood, then X is α-scattered.*

Proof. Assume that X is not α-scattered. Then the Cantor-Bendixson derivative $D(X)$ has a nonempty subset U which is open in X. Let $x \in U$. By assumption, there exists an α-scattered set V and an open set W such that $x \in W \subseteq V$. Since by Theorem 2.18, α-scatteredness is open hereditary, then W is α-scattered and so is $A = U \cap W$. Since A is nonempty and open, then A has an isolated point p. Clearly p must be open in X. Since $p \notin I(X)$, by contradiction X is α-scattered. $\Box$

¿From the above given proof it is clear that if every point of the Cantor-Bendixson derivative of a topological space X has an α-scattered neighborhood, then X itself is α-scattered.

Example 2.30 The union of even two α-scattered sets need not be α-scattered. Let $X = \{a, b, c, d\}$, where the non-trivial open sets are $\{a, b\}$ and $\{c, d\}$. Clearly $\{a, d\}$ and $\{b, c\}$ are α-scattered but their union is X fails to be α-scattered.

A topological space (X, τ) is called *sporadic* if the Cantor-Bendixson derivative of X is meager. Clearly every α-scattered space is sporadic and the space of all rationals (as a subspace of the real line) provides an example of a sporadic space which fails to be α-scattered.

Recall that a topological space (X, τ) is called *d-Lindelöf* [10] if every cover of X by dense subsets has a countable subcover.

Theorem 2.31 *Every d-Lindelöf, semi-$T_{\frac{1}{2}}$-space is sporadic.*

Proof. By Proposition 2.4 from [10], the Cantor-Bendixson derivative of X is countable. Since X is a semi-$T_{\frac{1}{2}}$-space, then for every $x \in D(X)$ we have $\{x\}$ is semi-closed. Hence, it is easily observed that each singleton of $D(X)$ is nowhere dense. Thus $D(X)$ is meager and consequently X is sporadic. $\Box$

None of the assumptions in the theorem above can be dropped, since one can easily find a d-Lindelöf or a semi-$T_{\frac{1}{2}}$-space that is not sporadic. Moreover, sporadic cannot be replaced by

α-scattered as the space of the rationals provides an example of a d-Lindelöf, semi-$T_{\frac{1}{2}}$-space which is not α-scattered.

Recall that a topological space (X, τ) is called *pseudo-Lindelöf* [3] if every uncountable subset of X has a limit point in X.

Corollary 2.32 *Every pseudo-Lindelöf submaximal space is sporadic and Lindelöf.*

Proof. Every pseudo-Lindelöf submaximal space is d-Lindelöf [3, Theorem 5.6]. Since X is submaximal, then X is semi-$T_{\frac{1}{2}}$. By Theorem 2.31, X is sporadic. By [3, Theorem 5.7], X is Lindelöf. □

Recall that a topological space (X, τ) is called *ω-scattered* [12] if every nonempty subset $A \subseteq X$ has a point x and an open neighborhood U of x in X such that $|U \cap A| \leq \aleph_0$.

Theorem 2.33 *Every d-Lindelöf space is ω-scattered.*

Proof. Let A be a nonempty subset of X. If $I = A \cap I(X) \neq \emptyset$, then every point of I satisfies the condition in the definition of ω-scatteredness. If A is placed in the Cantor-Bendixson derivative of X, then A is at most countable since X is d-Lindelöf. So clearly any neighborhood (in X) of a point of A meets A in a countable amount. Thus X is ω-scattered. □

It shown in [12] that every ω-scattered, compact, Hausdorff space is α-scattered.

Question 1. When is an ω-scattered space α-scattered? Can the assumption 'compact, Hausdorff' be reduced?

Question 2. When is a sporadic space α-scattered? When do the two concepts coincide?

Recall that a topological space (X, τ) is called *C-scattered* [29] if every closed subspace of X has a point with a compact neighborhood.

Question 3. How are α-scattered and C-scattered spaces related?

Recall that a topological space (X, τ) is called a *Luzin space* [19] if X satisfies the following four conditions: (1) X is Hausdorff; (2) X is uncountable; (3) $|I(X)| \leq \aleph_0$ and (4) $|N| \leq \aleph_0$ for each $N \in \mathcal{N}$. It is easily observed that sporadic Luzin spaces do not exist and that if a space X is Luzin, then X is not d-Lindelöf.

Recently there has been a significant interest in the study of rim-scattered space [8, 13, 14, 20]. Recall that a topological space (X, τ) is called *rim-scattered* if X has a basis of open sets with scattered boundaries. It is easy to find examples showing that rim-scattered and α-scattered spaces are independent concepts. It is also easy to see that a space is scattered iff it is α-scattered and N-scattered, i.e. every open set has scattered boundary [7, 23]. The following example will show that N-scatteredness is a property placed properly between scatteredness and rim-scatteredness.

Example 2.34 (1) Recall that a measurable set $E \subseteq \mathbf{R}$ has density d at $x \in \mathbf{R}$ if

$$\lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

exists and is equal to d. Set $\phi(E) = \{x \in \mathbf{R} \colon d(x, E) = 1\}$. The open sets of the density topology τ_d are those measurable sets E that satisfy $E \subseteq \phi(E)$. Note that every open set of the density topology has nowhere dense and hence closed and discrete boundary [28]. Thus the density topology is N-scattered. But the density topology is not even α-scattered; in fact it is crowded.

(2) Let $\mathbf{R}$ be the real line with the following topology: $\tau = \{A \subseteq \mathbf{R} \colon A \cap \{0,1\} = \emptyset\} \cup \{B \subseteq \mathbf{R} \colon \{0,1\} \subseteq B$ and $\mathbf{R} \setminus B$ is finite$\}$. Set $\mathcal{B} = \{\{x\} \colon x \neq 0$ and $x \neq 1\} \cup \{B \subseteq \mathbf{R} \colon \{0,1\} \subseteq B$ and $\mathbf{R} \setminus B$ is finite$\}$. Clearly $\mathcal{B}$ is a basis for τ consisting of clopen sets (so the space is zero-dimensional, i.e. its Menger-Urysohn dimension is at most 0). Hence $(\mathbf{R}, \tau)$ is rim-scattered. But the set $A = \mathbf{R} \setminus \{0,1\}$ is open and has non-scattered boundary. Thus $(\mathbf{R}, \tau)$ is not N-scattered. However this is an α-scattered space, which is not scattered. Hence α-scattered, rim-scattered spaces need not always be scattered.

The following results shows the connection between α-scattered and zero-dimensional spaces. An *Alexandroff space* (or also called *principal* space) is a space where arbitrary intersection of open sets is open.

Theorem 2.35 *An α-scattered, Alexandroff space (X, τ) is zero-dimensional if and only if X is discrete.*

Proof. Assume that X is nonempty. Since X is zero-dimensional, then every point of $I(X)$ is also closed in X. Note that $I(X) \neq \emptyset$, since X is α-scattered. Since X is Alexandroff, then the Cantor-Bendixson derivative $D(X)$ is clopen and $D(X) \neq X$. Since $I(X)$ is dense in X, then $D(X) = \emptyset$, i.e. X is discrete. $\square$

Recently Arhangel'skiĭ and Collins [3, Theorem 1.6] proved that a topological space (X, τ) is an I-space if and only if X is an α-space and α-scattered. Recall that a space (X, τ) is called an *I-space* [3] if the Cantor-Bendixson derivative of X is closed and discrete. Next we will improve the above mentioned result of Arhangel'skiĭ and Collins but in order to do that we define a topological space X to be an *α_i-space* if every α-open subspace of X which satisfies the separation axiom T_i is open in X. In [3, Theorem 1.6] the term nodec was used instead of α-space but it is known that a space is nodec if and only if it is an α-space [6]. Recall that a topological space (X, τ) is called a *$T_{\frac{1}{2}}$-space* if every non-closed singleton is isolated.

Theorem 2.36 *For a topological space (X, τ) the following conditions are equivalent:*

(1) *X is an I-space.*

(2) *X is α-scattered and an $\alpha_{\frac{1}{2}}$-space.*

Proof. (1) $\Rightarrow$ (2) follows from [3, Theorem 1.6].

(2) $\Rightarrow$ (1) For each point x of the Cantor-Bendixson derivative $D(X)$, let $A_x = I(X) \cup \{x\}$. Note first that A_i is α-open in X. Moreover it is easily observed that A_x is a $T_{\frac{1}{2}}$-space. Since X is an $\alpha_{\frac{1}{2}}$-space, then each A_x is open in X. Hence each $x \in D(X)$ is open in $D(X)$ and thus X is an I-space. $\square$

The following corollary improves Corollary 1.8 from [3] and Theorem 3 from [23].

Corollary 2.37 *In the class of α-scattered spaces the notions of an I-space, of a nodec space, of a submaximal space and of a $\alpha_{\frac{1}{2}}$-space are pairwise equivalent.* $\square$

Example 2.38 (1) Note that $\alpha_{\frac{1}{2}}$-space is a concept strictly weaker than the one of an α-space. Consider for instance $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a, b\}, X\}$.

(2) In [3], Arhangel'skiĭ and Collins introduced the notion of a discretely flavoured space. By definition a topological space (X, τ) is called *discretely flavoured* [3] if every infinite closed crowded subspace contains a discrete non-closed set. Every scattered space is discretely flavoured [3, Proposition 3.3 (a)]. Here is an example of an α-scattered space that fails to be discretely flavoured: Let X be the real line with the following topology: $\tau = \{\emptyset, X, \{0\}\} \cup \{A \subseteq X : 0 \in A \text{ and } X \setminus A \text{ is finite}\}$. Having the zero-point as a generic point, this space is obviously α-scattered. But note that $X \setminus \{0\}$ is infinite, closed and crowded subspace inheriting the cofinite topology whose only discrete sets are the closed ones.

In the same paper [3], Arhangel'skiĭ and Collins introduced the concept of a *j-space*, i.e. of a space where every infinite closed crowded subspace contains an infinite subset which is countably compact in X. They proved [3, Proposition 6.7] that a Hausdorff α-space is a j-space if and only if it is scattered. We offer a slight improvement of their result.

Theorem 2.39 *For a Hausdorff α-space (X, τ) the following conditions are equivalent:*

(1) *X is a j-space.*

(2) *X is α-scattered.* $\square$

A classical result on nowhere dense sets is the fact that in crowded metric spaces discrete sets have nowhere dense closures [1, 31]. We offer the following improvement of this result:

Theorem 2.40 *In crowded $T_{\frac{1}{2}}$-spaces, α-scattered sets have nowhere dense closures.*

Proof. Let (X, τ) be crowded and $T_{\frac{1}{2}}$ (and hence T_1).

Step 1. We show first that the closure of every discrete subset A of X is nowhere dense. Let U be an nonempty open subset of $\overline{A}$ and let $x \in U \cap A$. Since A is discrete, then there exists $V \in \tau$ such that $V \cap A = \{x\}$. Note that $|U \cap V| > 1$, since X is crowded. Let $y \in U \cap V$ with $x \neq y$. Since X is T_1, then there exists an open set W containing y such that $x \notin W$. Now $U \cap V \cap W \subseteq \overline{A}$ is nonempty, open and disjoint from A. By contradiction, $\overline{A}$ is codense, i.e. the closure of A is nowhere dense in X.

Step 2. Let A be an α-scattered subset of X. Clearly $I(A)$ is a discrete subset of X and by Step 1, $\overline{I(A)}$ is nowhere dense. On the other hand, $D(A)$ is nowhere dense in A, since A is α-scattered. Thus $D(A)$ is a nowhere dense subset of X and hence $\overline{D(A)}$ is nowhere dense. Consequently $\overline{A}$ is nowhere dense being the finite union of nowhere dense sets. $\square$

3 π-continuous and θ-continuous functions

In [23], a function $f\colon (X,\tau) \to (Y,\sigma)$ is called *π-continuous* if τ is a π-base for the weak topology $f^{-1}(\sigma) = \{f^{-1}(V)\colon V \in \sigma\}$. π-continuity can be characterized as follows:

Theorem 3.1 *For a function $f\colon X \to Y$, the following conditions are equivalent:*

(1) *f is π-continuous.*

(2) *If D is dense in X, then $f(D)$ is dense in $f(X)$.* $\square$

Note that π-continuous surjections are usually called *somewhat continuous.*

Proposition 3.2 *If X is α-scattered, then every function $f\colon X \to Y$ has the property of Baire, i.e., inverse images of open sets have the property of Baire.*

It is also clear that α-scattered spaces are Baire spaces since $\mathcal{M}(\tau) = \mathcal{N}(\tau)$ is always τ-codense. Recall that a set A is called *almost open* (classically) if A is contained in the interior of $A^*(\mathcal{M}(\tau))$. ¿From [30, p. 298] (see also [11]), $A^*(\mathcal{N}(\tau)) = \mathrm{Cl}\mathrm{Int}\mathrm{Cl}A$. Thus, a subset A of an α-scattered space is almost open if and only if A is contained in $\mathrm{Int}\mathrm{Cl}A$, i.e., A is nearly open (also called preopen). By an extension of M. Wilhelm's Theorem: (see [22, 25]) we have the following.

Proposition 3.3 *If $f\colon X \to Y$ is nearly continuous (sometimes called precontinuous) and X is α-scattered, then f is θ-continuous and also, (as a consequence of Theorem 6.11 of [11] since $\mathcal{N}(\tau)$ is τ-codense; or [25, Lemma 7]) $f\colon X^\alpha \to Y$ is θ-continuous.* $\square$

Conditions (ii) and (iii) of the corollary below give necessary conditions for a nearly continuous function to have θ-closed and δ-closed graph, respectively. Note that δ-closed graphs (resp. θ-closed graphs) were considered by Cammaroto and Noiri in [5, Section C: δ-closed graphs] (resp. by Janković and Long in [15, Theorem 8]).

Corollary 3.4 (i) *Let (X,τ) be α-scattered. If there exists an Urysohn space (Y,σ) and a nearly continuous injective function $f\colon (X,\tau) \to (Y,\sigma)$, then X is also Urysohn.*

(ii) *If* (X, τ) *is* α*-scattered and* (Y, σ) *is Urysohn, then the graph of every nearly continuous function* $f : (X, \tau) \to (Y, \sigma)$ *is* θ*-closed.*

(iii) *If* (X, τ) *is* α*-scattered and* (Y, σ) *is Hausdorff, then the graph of every nearly continuous function* $f : (X, \tau) \to (Y, \sigma)$ *is* δ*-closed, i.e. the graph is closed in the semi-regularization topology of* $X \times Y$*.*

(iv) *If* (X, τ) *is an* α*-scattered, H-closed space, then every nearly continuous function* $f : (X, \tau) \to (Y, \sigma)$ *is almost closed given* (Y, σ) *is Hausdorff.*

(v) *If* (X, τ) *is* α*-scattered, then every nearly continuous function* $f : (X, \tau) \to (Y, \sigma)$ *is faintly* α*-continuous.* $\Box$

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