ON THE PRODUCT OF LC-SPACES

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Abstract

A topological space X is called an LC-space if every Lindelöf subset of X is closed. In this note we expand a result of Hdeib and Pareek by proving that the product of two Hausdorff LC-spaces is an LC-space . Moreover, we show that in general the product of an LC-space with itself need not be an LC-space.

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For a subset A of a topological space X we will denote the closure of A by $cl\ A$. For two spaces X and Y , $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ will denote the canonical projections. The set of positive integers is denoted by ω.

A space X is called an LC-space (or L-closed [2]) if every Lindelöf subset of X is closed. Note that every LC-space is a KC-space [4], i.e. a space in which compact subsets are closed. Familiar examples of LC-spaces are Hausdorff P-spaces [2] and maximal Lindelöf spaces [1] . Clearly, every LC-space is cid [3] , i.e. every countable subspace is closed and discrete, and thus T_1 . In [2] , Hdeib and Pareek obtained the following result of which we will include a short proof.

Theorem 1 [2] If X and Y are LC-spaces and either X or Y is regular, then $X \times Y$ is an LC-space.

Proof. Suppose that X is regular. Let L be a Lindelöf subset of $X \times Y$ and let $(x_0, y_0) \notin L$. If $L_1 = (X \times \{y_0\}) \cap L$, then L_1 is a Lindelöf subset of $X \times Y$ and $x_0 \notin \pi_1(L_1)$. Clearly $\pi_1(L_1))$ is a Lindelöf subset of X and so there exists an open neighborhood $G \subseteq X$ of x_0 such that $cl\ G \cap \pi_1(L_1) = \emptyset$. If $L_2 = (cl\ G \times Y) \cap L$ then L_2 is a Lindelöf subset of $X \times Y$ and $y_0 \notin \pi_2(L_2)$. Since $\pi_2(L_2)$ is closed in Y, there exists an open neighborhood

$H \subseteq Y$ of y_0 such that $H \cap \pi_2(L_2) = \emptyset$. One easily checks that $(G \times H) \cap L = \emptyset$ and thus $X \times Y$ is an LC-space. ♣

In the following we will address the question whether this result holds for general LC-spaces. We will show that the product of two Hausdorff LC-spaces is again an LC-space. However, there exists an LC-space whose product with itself fails to be an LC-space. So the question above is resolved completely.

Theorem 2 If X and Y are Hausdorff LC-spaces, then $X \times$ is an LC-space.

Proof. Let L be a Lindelöf subset of $X \times Y$ and let $(x_0, y_0) \notin L$. For each $(x, y) \in L$ there exist open neighborhoods $U_x \subseteq X$ of x and $V_y \subseteq Y$ of y such that $(x_0, y_0) \notin cl\ U_x \times cl\ V_y$.

Since $L \subseteq \bigcup\{\ U_x \times V_y\ :\ (x, y) \in L\}$ we have $L \subseteq \bigcup\{\ U_{x_n} \times V_{y_n}\ :\ n \in \omega\}$ for some $(x_n, y_n) \in L,\ n \in \omega$. Now let $E_1 = \{n \in \omega\ :\ x_0 \notin cl\ U_{x_n}\}$ and $E_2 = \{n \in \omega\ :\ y_0 \notin cl\ V_{y_n}\}$. Then $E_1 \cup E_2 = \omega$, and if $L_1 = \bigcup\{\ L \cap (cl\ U_{x_n} \times cl\ V_{y_n})\ :\ n \in E_1\}$ and $L_2 = \bigcup\{\ L \cap (cl\ U_{x_n} \times cl\ V_{y_n})\ :\ n \in E_2\}$ then L_1 and L_2 are Lindelöf subsets of $X \times Y$ such that $L_1 \cup L_2 = L$. Clearly, $x_0 \notin \pi_1(L_1)$ and so there is an open neighborhood $G \subseteq X$ of x_0 with $G \cap \pi_1(L_1) = \emptyset$. In the same manner, since $y_0 \notin \pi_2(L_2)$, we obtain an open neighborhood $H \subseteq Y$ of y_0 with $H \cap \pi_2(L_2) = \emptyset$. We now claim that $(G \times H) \cap L = \emptyset$. If $(x, y) \in G \times H$ then $x \notin \pi_1(L_1)$ and so $(x, y) \notin L_1$. Also, $y \notin \pi_2(L_2)$ and so $(x, y) \notin L_2$, i.e. $(x, y) \notin L$. This proves that $X \times Y$ is an LC-space. ♣

Recall that a space X is said to be R_1 if x and y have disjoint neighborhoods whenever $cl\ \{x\} \neq cl\ \{y\}$. Clearly, a space is Hausdorff if and only if it is T_1 and R_1 , and so we have

Corollary 3 If X and Y are R_1 LC- spaces then $X \times Y$ is an LC-space.

Theorem 4 There exists an LC-space X such that $X \times X$ is not an LC-space.

Proof. In [1] , Example 7.3, Cameron provides an example of a maximal Lindelöf space X which is not Hausdorff. Clearly X is an LC-space and also a P-space, i.e. G_δ-subsets are open. The diagonal $\Delta = \{(x, x)\ :\ x \in X\}$ is homeomorphic to X, and therefore a Lindelöf subset of $X \times X$. However, Δ is not closed, since X is not Hausdorff. So $X \times X$ is not an LC-space (although it is a P-space) . ♣

References

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