More on sg-compact spaces*

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Abstract

The aim of this paper is to continue the study of sg-compact spaces, a topological notion much stronger than hereditary compactness. We investigate the relations between sg-compact and C_2-spaces and the interrelations to hereditarily sg-closed sets.

1 Introduction

In 1995, sg-compact spaces were introduced independently by Caldas [2] and by Devi, Balachandran and Maki [4]. A topological space (X, τ) is called *sg-compact* [2] if every cover of X by sg-open sets has a finite subcover. In [4], the term *SGO-compact* is used.

Recall that a subset A of a topological space (X, τ) is called *sg-open* [1] if every semi-closed subset of A is included in the semi-interior of A. A set A is called *semi-open* if $A \subseteq \overline{\text{Int}A}$ and *semi-closed* if $\text{Int}\overline{A} \subseteq A$. The *semi-interior* of A, denoted by sInt(A), is the union of all semi-open subsets of A while the *semi-closure* of A, denoted by sCl(A), is the intersection of all semi-closed supersets of A. It is well known that $\text{sInt}(A) = A \cap \overline{\text{Int}A}$ and $\text{sCl}(A) = A \cup \text{Int}\overline{A}$.

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Every topological space (X,τ) has a unique decomposition into two sets X_1 and X_2, where $X_1 = \{x \in X : \{x\}$ is nowhere dense$\}$ and $X_2 = \{x \in X : \{x\}$ is locally dense$\}$. This decomposition follows from a result of Janković and Reilly [13, Lemma 2]. Recall that a set A is said to be *locally dense* [3] (= *preopen*) if $A \subseteq \mathrm{Int}\overline{A}$.

It is a fact that a subset A of X is sg-closed (= its complement is sg-open) if and only if $X_1 \cap \mathrm{sCl}(A) \subseteq A$ [6], or equivalently if and only if $X_1 \cap \mathrm{Int}\overline{A} \subseteq A$. By taking complements one easily observes that A is sg-open if and only if $A \cap X_1 \subseteq \mathrm{sInt}(A)$. Hence every subset of X_2 is sg-open.

2 Sg-compact spaces

Let A be a sg-closed subset of a topological space (X,τ). If every subset of A is also sg-closed in (X,τ), then A will be called *hereditarily sg-closed* (= hsg-closed). Observe that every nowhere dense subset is hsg-closed but not vice versa.

Proposition 2.1 *For a subset A of a topological space (X,τ) the following conditions are equivalent:*

(1) *A is hsg-closed.*

(2) $X_1 \cap \mathrm{Int}\overline{A} = \emptyset$.

Proof. (1) $\Rightarrow$ (2) Suppose that there exits $x \in X_1 \cap \mathrm{Int}\overline{A}$. Let V_x be an open set such that $V_x \subseteq \overline{A}$ and let $B = A \setminus \{x\}$. Since B is sg-closed, i.e. $X_1 \cap \mathrm{sCl}(B) \subseteq B$, we have $x \notin \mathrm{sCl}(B)$, hence $x \notin \mathrm{Int}\overline{B}$, and thus $x \in \overline{X \setminus \overline{B}}$. If $H = V_x \cap (X \setminus \overline{B})$, then H is nonempty and open with $H \subseteq \overline{A}$ and $H \cap B = \emptyset$ and so $H \cap A = \{x\}$. Hence $\emptyset \neq H = H \cap \overline{A} \subseteq \overline{H \cap A} \subseteq \overline{\{x\}}$, i.e. $\mathrm{Int}\overline{\{x\}} \neq \emptyset$. Thus $x \in X_2$, a contradiction.

(2) $\Rightarrow$ (1) Let $B \subseteq A$. Then $\mathrm{Int}\overline{B} \subseteq \mathrm{Int}\overline{A}$ and $X_1 \cap \mathrm{Int}\overline{B} = \emptyset$, i.e. B is sg-closed. □

We will call a topological space (X,τ) a *C_2-space* [9] (resp. *C_3-space*) if every nowhere dense (resp. hsg-closed) set is finite. Clearly every C_3-space is a C_2-space. Also, a topological space (X, τ) is indiscrete if and only if every subset of X is hsg-closed (since in that case $X_1 = \emptyset$).

Following Hodel [14], we say that a *cellular family* in a topological space (X, τ) is a collection of nonempty, pairwise disjoint open sets. The following result reveals an interesting property of C_2-spaces.

Lemma 2.2 *Let (X, τ) be a C_2-space. Then every infinite cellular family has an infinite subfamily whose union is contained in X_2.*

Proof. Let $\{U_i : i \in \mathbf{N}\}$ be a cellular family. Suppose that for infinitely many $i \in \mathbf{N}$ we have $U_i \cap X_1 \neq \emptyset$. Without loss of generality we may assume that $U_i \cap X_1 \neq \emptyset$ for each $i \in \mathbf{N}$. Now pick $x_i \in U_i \cap X_1$ for each $i \in \mathbf{N}$ and partition $\mathbf{N}$ into infinitely many disjoint infinite sets, $\mathbf{N} = \cup_{k \in \mathbf{N}} \mathbf{N}_k$. Let $A_k = \{x_i : i \in \mathbf{N}_k\}$. Since $A_k \cap (\cup_{i \notin \mathbf{N}_k} U_i) = \emptyset$ and $A_k \subseteq \cup_{i \in \mathbf{N}_k} U_i$ for each k, it is easily checked that $\{\mathrm{Int}\overline{A_k} : k \in \mathbf{N}\}$ is a disjoint family of open sets. Since X is a C_2-space, A_k cannot be nowhere dense and so, for each k, there exists $p_k \in \mathrm{Int}\overline{A_k}$ and the p_k's are pairwise distinct. Also, since X is C_2, $\overline{\cup_{i \in \mathbf{N}} U_i} = \cup_{i \in \mathbf{N}} (U_i) \cup F$, where F is finite. Since $p_k \in \overline{\cup_{i \in \mathbf{N}} U_i}$ for each k, there exists k_0 such that $p_k \in \cup_{i \in \mathbf{N}} U_i$ for $k \geq k_0$, and since $\mathrm{Int}\overline{A_k} \cap (\cup_{i \notin \mathbf{N}_k} U_i) = \emptyset$, we have $p_k \in \cup_{i \in \mathbf{N}_k} U_i$ for $k \geq k_0$. Now, for each $k \geq k_0$ pick $i_k \in \mathbf{N}_k$ such that $p_k \in U_{i_k}$, and so $p_k \in W = U_{i_k} \cap \mathrm{Int}\overline{A_k}$. Thus $\emptyset \neq W \subseteq U_{i_k} \cap \overline{A_k} \subseteq \overline{U_{i_k} \cap A_k} = \overline{\{x_{i_k}\}}$. Hence $\{x_{i_k}\}$ is locally dense, a contradiction. This shows that only for finitely many $i \in \mathbf{N}$ we have $U_i \cap X_1 \neq \emptyset$. Thus the claim is proved. □

The *α-topology* [16] on a topological space (X, τ) is the collection of all sets of the form $U \setminus N$, where $U \in \tau$ and N is nowhere dense in (X, τ). Recall that topological spaces whose α-topologies are hereditarily compact have been shown to be *semi-compact* [11]. The original definition of semi-compactness is in terms of semi-open sets and is due to Dorsett [8]. By definition a topological space (X, τ) is called *semi-compact* [8] if every cover of X by semi-open sets has a finite subcover.

Remark 2.3 (i) The 1-point-compactification of an infinite discrete space is a C_2-space having an infinite cellular family.

(ii) [9] A topological space (X, τ) is semi-compact if and only if X is a C_2-space and every cellular family is finite.

(iii) [12] Every subspace of a semi-compact space is semi-compact (as a subspace).

Lemma 2.4 (i) *Every C_3-space (X, τ) is semi-compact.*

(ii) *Every sg-compact space is semi-compact.*

Proof. (i) All C_3-spaces are C_2-spaces. Thus in the notion of Remark 2.3 (ii) above we need to show that every cellular family in X is finite. Suppose that there exists an infinite cellular family $\{U_i : i \in \mathbf{N}\}$. For each $i \in \mathbf{N}$ pick $x_i \in U_i$ and, as before, partition $\mathbf{N} = \cup_k \mathbf{N}_k$ and set $A_k = \{x_i : i \in \mathbf{N}_k\}$. Since X is a C_2-space, $\{\mathrm{Int}\overline{A_k} : k \in \mathbf{N}\}$ is a cellular family. By Lemma 2.2, there is a $k \in \mathbf{N}$ such that $\mathrm{Int}\overline{A_k} \subseteq X_2$. Since A_k is not hsg-closed, we must have $X_1 \cap \mathrm{Int}\overline{A_k} \neq \emptyset$, a contradiction. So, every cellular family in X is finite and consequently (X, τ) is semi-compact.

(ii) is obvious since every semi-open set is sg-open. □

Remark 2.5 (i) It is known that sg-open sets are β-open, i.e. they are open in some regular closed subspace [5]. Note that β-compact spaces, i.e. the spaces in which every cover by β-open sets has a finite subcover are finite [10]. However, one can easily find an example of an infinite sg-compact space – the real line with the cofinite topology is such a space.

(ii) In semi-T_D-spaces the concepts of sg-compactness and semi-compactness coincide. Recall that a topological space (X, τ) is called a *semi-T_D-space* [13] if each singleton is either open or nowhere dense, i.e. if every sg-closed set is semi-closed.

Theorem 2.6 *For a topological space (X, τ) the following conditions are equivalent:*

(1) *X is sg-compact.*

(2) *X is a C_3-space.*

Proof. (1) $\Rightarrow$ (2) Suppose that there exists an infinite hsg-closed set A and set $B = X \setminus A$. Observe that for each $x \in A$, the set $B \cup \{x\}$ is sg-open in X. Thus $\{B \cup \{x\} : x \in A\}$ is a sg-open cover of X with no finite subcover. Thus (X, τ) is C_3.

(2) $\Rightarrow$ (1) Let $X = \cup_{i\in I} A_i$, where each A_i is sg-open. Let $S_i = \text{sInt}(A_i)$ for each $i \in I$ and let $S = \cup_{i\in I} S_i$. Then S is a semi-open subset of X and each S_i is a semi-open subset of $(S, \tau|S)$. Since X is a C_3-space, (X, τ) is semi-compact and hence $(S, \tau|S)$ is a semi-compact subspace of X (by Remark 2.3 (iii)). So we may say that $S = S_{i_1} \cup \ldots \cup S_{i_k}$. Since A_i is sg-open, we have $X_1 \cap A_i \subseteq S_i$ for each index i and so $X_1 = X_1 \cap (\cup A_i) \subseteq X_1 \cap S \subseteq S_{i_1} \cup \ldots \cup S_{i_k} = S$. Hence $X \setminus S$ is semi-closed and $X \setminus S \subseteq X_2$. Since $\text{Int}\overline{(X \setminus S)} \subseteq X \setminus S \subseteq X_2$, we conclude that $X \setminus S$ is hsg-closed and thus finite. This shows that $X = S_{i_1} \cup \ldots \cup S_{i_k} \cup (X \setminus S) = A_{i_1} \cup \ldots \cup A_{i_k} \cup F$, where F is finite, i.e. (X, τ) is sg-compact. $\square$

Remark 2.7 (i) If $X_1 = X$, then (X, τ) is sg-compact if and only if (X, τ) is semi-compact. Observe that in this case sg-closedness and semi-closedness coincide.

(ii) Every infinite set endowed with the cofinite topology is (hereditarily) sg-compact.

It is known that an arbitrary intersection of sg-closed sets is also an sg-closed set [6]. The following result provides an answer to the question about the additivity of sg-closed sets.

Proposition 2.8 (i) *If A is sg-closed and B is closed, then $A \cup B$ is also sg-closed.*

(ii) *The intersection of a sg-open and an open set is always sg-open.*

(iii) *The union of a sg-closed and a semi-closed set need not be sg-closed, in particular, even finite union of sg-closed sets need not be sg-closed.*

Proof. (i) Let $A \cup B \subseteq U$, where U is semi-open. Since A is sg-closed, we have $\text{sCl}(A \cup B) = (A \cup B) \cup \text{Int}(\overline{A \cup B}) \subseteq U \cup \text{Int}(\overline{A} \cup B) \subseteq U \cup (\text{Int}\overline{A} \cup B) \subseteq U \cup (U \cup B) = U$.

(ii) follows from (i).

(iii) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Note that the two sets $A = \{a\}$ and $B = \{b\}$ are semi-closed but their union $\{a, b\}$ is not sg-closed. $\square$

Theorem 3 from [1] states that if $B \subseteq A \subseteq (X, \tau)$ and A is open and sg-closed, then B is sg-closed in the subspace A if and only if B is sg-closed in X. Since a subset is regular open if and only if it is α-open and sg-closed [7], by using Proposition 2.8, we obtain the following result:

Proposition 2.9 *Let R be a regular open subset of a topological space (X, τ). If $A \subseteq R$ and A is sg-open in $(R, \tau|R)$, then A is sg-open in X.* $\square$

Proof. Since $B = R \setminus A$ is sg-closed in $(R, \tau|R)$, B is sg-closed in X by [1, Theorem 3]. Thus $X \setminus B$ is sg-open in X and by Proposition 2.8 (ii), $R \cap (X \setminus B) = A$ is sg-open in X. $\square$

Recall that a subset A of a topological space (X, τ) is called *δ-open* [18] if A is a union of regular open sets. The collection of all δ-open subsets of a topological space (X, τ) forms the so called *semi-regularization topology.*

Corollary 2.10 *If $A \subseteq B \subseteq (X, \tau)$ such that B is δ-open in X and A is sg-open in B, then A is sg-open in X.*

Proof. Let $B = \cup_{i \in I} B_i$, where each B_i is regular open in (X, τ). Clearly, each B_i is regular open also in $(B, \tau|B)$. By Proposition 2.8 (ii), $A \cap B_i$ is sg-open in $(B, \tau|B)$ for each $i \in I$. In the notion of Proposition 2.9, $B \setminus (A \cap B_i)$ is sg-closed in (X, τ) for each $i \in I$. Hence $X \setminus (B \setminus (A \cap B_i)) = (A \cap B_i) \cup (X \setminus B)$ is sg-open in (X, τ). Again by Proposition 2.8 (ii), $B \cap ((A \cap B_i) \cup (X \setminus B)) = A \cap B_i$ is sg-open in (X, τ). Since any union of sg-open sets is always sg-open, we have $A = \cup_{i \in I} (A \cap B_i)$ is sg-open in (X, τ). $\square$

Proposition 2.11 *Every δ-open subset of a sg-compact space (X, τ) is sg-compact, in particular, sg-compactness is hereditary with respect to regular open sets.*

Proof. Let $A \subseteq X$ be δ-open. If $\{U_i : i \in I\}$ is a sg-open cover of $(S, \tau|S)$, then by Corollary 2.10, each U_i is sg-open in X. Then, $\{U_i : i \in I\}$ along with $X \setminus A$ forms a sg-open cover of X. Since X is sg-compact, there exists a finite $F \subseteq I$ such that $\{U_i : i \in F\}$ covers A. $\square$

Example 2.12 Let A be an infinite set with $p \notin A$. Let $X = A \cup \{p\}$ and $\tau = \{\emptyset, A, X\}$.

(i) Clearly, $X_1 = \{p\}$, $X_2 = A$ and for each infinite $B \subseteq X$, we have $\overline{B} = X$. Hence $X_1 \cap \text{Int}\overline{B} \neq \emptyset$, so B is not hsg-closed. Thus (X, τ) is a C_3-space, so sg-compact. But the open subspace A is an infinite indiscrete space which is not sg-compact. This shows that (1) hereditary sg-compactness is a strictly stronger concept than sg-compactness and (2) in Proposition 2.11 'δ-open' cannot be replaced with 'open'.

(ii) Observe that $X \times X$ contains an infinite nowhere dense subset, namely $X \times X \setminus A \times A$. This shows that even the finite product of two sg-compact spaces need not be sg-compact, not even a C_2-space.

(iii) [15] If the nonempty product of two spaces is sg-compact T_{gs}-space (see [15]), then each factor space is sg-compact.

Recall that a function $f\colon (X, \tau) \to (Y, \sigma)$ is called *pre-sg-continuous* [17] if $f^{-1}(F)$ is sg-closed in X for every semi-closed subset $F \subseteq Y$.

Proposition 2.13 (i) *The property 'sg-compact' is topological.*

(ii) *Pre-sg-continuous images of sg-compact spaces are semi-compact.* □

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