AN ANSWER TO A QUESTION OF DORSETT

Maximilian GANSTER[1] and Alev KANIBIR[2]

Abstract

We show that the topology generated by the semi-open sets of a given space need not be extremally disconnected. This answers a recent question of Charles Dorsett.

[1] Department of Mathematics, Graz University of Technology, Graz, AUSTRIA
[2)] Department of Mathematics, Zonguldak Karaelmas University, Zonguldak, TURKEY

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For a subset A of a topological space (X, τ) the closure of A and the interior of A will be denoted by $cl_\tau A$ and $int_\tau A$, respectively. A subset S of (X, τ) is called *semi-open* [4] if there exists an open set U such that $U \subseteq S \subseteq cl_\tau U$. We will denote the family of all semi-open subsets of (X, τ) by $SO(X, \tau)$. For $A \subseteq X$, the semi-interior of A is the largest semi-open set contained in A and is denoted by $sint_\tau A$. It is well known that $sint_\tau A = A \cap cl_\tau(int_\tau A)$. Following Eric van Douwen [2] , $x \in X$ will be called an *e.d.-point* of (X, τ) if for every τ-open set U , if $x \in cl_\tau U$ then $x \in int_\tau(cl_\tau U)$. Otherwise we say that $x \in X$ is a *non-e.d.-point* of (X, τ) . It is clear that a space (X, τ) is extremally disconnected, i.e. the closure of every open set is open, if and only if every point of X is an e.d.-point .

In [1] Dorsett asked the following

Question. For a space (X, τ) let σ be the topology on X having $SO(X, \tau)$ as a subbase. Is (X, σ) extremally disconnected ?

In our note we will provide a negative answer to this question. Also, there will be some overlap between our discussion and some results in [1] . We think, however, that our presentation makes these results in [1] more accessible to the reader.

For the following let (X, τ) be any topological space. Then $X = A \cup B$ where A (resp. B) denotes the set of e.d.-points (resp. non-e.d.-points) of (X, τ) . The topology having $SO(X, \tau)$ as a subbase is denoted by σ .

Observation 1 [1] If $x \in B$ then $\{x\} \in \sigma$.

Proof. Let $x \in B$. Then there is a τ-open set U with $x \in cl_\tau U \setminus int_\tau(cl_\tau U)$. If $S_1 = U \cup \{x\}$ and $S_2 = (X \setminus cl_\tau U) \cup \{x\}$, then $S_1, S_2 \in SO(X, \tau)$ and $S_1 \cap S_2 = \{x\}$, hence $x \in \sigma$. $\square$

Theorem 2 Let $W \subseteq X$. Then $int_\sigma W = (W \cap B) \cup (A \cap sint_\tau W)$.

Proof. If $x \in W \cap B$, then $\{x\} \in \sigma$ and so $x \in int_\sigma W$. If $x \in A \cap sint_\tau W$, then again $x \in int_\sigma W$, since $sint_\tau W \in SO(X, \tau) \subseteq \sigma$.

Conversely, let $x \in int_\sigma W$. If $x \in B$ then $x \in W \cap B$. Now let $x \in A$. Then there exist $S_1, S_2, ...S_k \in SO(X, \tau)$ such that $x \in S_1 \cap S_2 \cap ... \cap S_k \subseteq W$. For each i , let U_i be τ-open with $U_i \subseteq S_i \subseteq cl_\tau U_i$. Since $x \in A$, we have $x \in int_\tau(cl_\tau U_i)$ for each i , and so there exists a τ-open set H containing x with $H \subseteq int_\tau(cl_\tau U_i)$ for each i . Suppose that $x \notin sint_\tau W$. Then $x \notin cl_\tau(int_\tau W)$, and so there exists a τ-open set G containing x with $G \cap int_\tau W = \emptyset$. Consequently $G \cap U_1 \cap ... \cap U_k = \emptyset$, and thus $G \cap int_\tau(cl_\tau U_1) \cap ... \cap int_\tau(cl_\tau U_k) = \emptyset$. But then $G \cap H$ has to be empty, a contradiction. So $x \in sint_\tau W$, and we are done. $\square$

By passing on to complements one easily proves

Theorem 3 Let $W \subseteq X$. Then $cl_\sigma W = W \cup (A \cap int_\tau(cl_\tau W))$.

This result suggests a way of finding spaces (X, τ) such that (X, σ) fails to be extremally disconnected.

Theorem 4 Let (X, τ) be a space where A and B are nonempty. If there is a decomposition of B into two disjoint τ-dense subsets C and D , then (X, σ) is not extremally disconnected.

Proof. Let $B = C \cup D$ with $C \cap D = \emptyset$ and $cl_\tau C = cl_\tau D = X$. Now consider $C \subseteq X$. Then $C \in \sigma$ by Observation 1 , and $cl_\sigma C = C \cup A$ by Theorem 3 . Since $cl_\tau D = X$, we have $int_\tau(C \cup A) = \emptyset$, and so $sint_\tau(C \cup A) = \emptyset$. Hence, if $x \in A$ then $x \in cl_\sigma C$ but $x \notin int_\sigma(cl_\sigma C)$ by Theorem 2 . So (X, σ) is not extremally disconnected. □

Fortunately, Eric van Douwen [3] has proven the existence of spaces satisfying the hypothesis of Theorem 4.

Example 5 (see [3] , page 26, Example 10.4)

Let $\mathbb{Q}$ denote the space of rationals and $\beta\mathbb{Q}$ its Stone-Cech-compactification. In [3] , van Douwen has pointed out that there is a point $q \in \beta\mathbb{Q} \setminus \mathbb{Q}$ such that q is the only e.d.-point of the subspace $X = \mathbb{Q} \cup \{q\}$. Let τ denote the topology on X . So, in our notation, we have $A = \{q\}$ and $B = \mathbb{Q}$. It is well known that $\mathbb{Q}$ has two disjoint dense subsets, say C and D , which clearly are also dense in (X, τ) . By Theorem 4 , the space (X, σ) where σ denotes the topology on X generated by $SO(X, \tau)$ cannot be extremally disconnected. In addition, $\{q\}$ is the only non-e.d.-point of (X, σ) and each $x \in \mathbb{Q}$ is isolated in (X, σ) . By Observation 1 , if ρ denotes the topology on X having $SO(X, \sigma)$ as a subbase, then (X, ρ) is discrete and so $\sigma \neq \rho$. This answers another question of Dorsett in [1] . □

Thus the topology generated by the semi-open sets of a given space need not be extremally disconnected.

References

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