# Ideal Resolvability*

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**Abstract**

A nonempty topological space is resolvable if it contains complementary dense subsets. The aim of this paper is to study resolvability modulo an ideal and to prove that the density topology is resolvable.

## 1 Introduction

In 1943, Hewitt introduced the concept of a resolvable space. By definition, a nonempty topological space $(X, \tau)$ is called *resolvable* [26] if $X$ is the disjoint union of two dense (or equivalently codense) subsets. In the opposite case $X$ is called *irresolvable*. Every space $(X, \tau)$ has its unique Hewitt representation, i.e. $X = F \cup G$, where $F$ is closed and resolvable, $G$ is hereditarily irresolvable and $F \cap G = \emptyset$ [26].

Hewitt [26] proved that every locally compact dense-in-itself Hausdorff space is resolvable and that every metrizable dense-in-itself space is resolvable. In particular, the Cantor subspace of $[0, 1]$ is resolvable. In 1987, Ganster [16] showed that a connected space $(X, \tau)$ is resolvable if and only if the topology on $X$ having the preopen sets of $(X, \tau)$ as a subbase is the discrete one. In 1988, P. Sharma and S. Sharma [45] improved Hewitt's result by proving that every Hausdorff $k$-space without isolated points is resolvable. In 1993, Comfort and Feng [7] proved that every homogeneous space with a nonempty resolvable subspace is resolvable. Also, all homogeneous spaces containing convergent sequences are resolvable

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[47]. It is well-known that a space $(X, \tau)$ is resolvable if and only if $X$ is a finite union of codense sets [4, 27]. Also, any resolvable space is dense-in-itself and resolvability is preserved by (semi-)open subspaces. For an example of a connected, Hausdorff, irresolvable space see [1, 39]. Resolvability of topological groups was recently studied by Comfort and van Mill [9] and by Comfort, Masaveu and Zhou [10]. An extensive survey on resolvability was recently made by Comfort and García-Ferreira [8].

A nonempty collection $\mathcal{I}$ of subsets on a topological space $(X, \tau)$ is called a *topological ideal* on $(X, \tau)$ if it satisfies the following two conditions:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (heredity).

(2) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

A *$\sigma$-ideal* on a topological space $(X, \tau)$ is a topological ideal which also satisfies the following condition:

(3) If $\{A_i : i = 1, 2, 3, \ldots\} \subseteq \mathcal{I}$, then $\cup\{A_i : i = 1, 2, 3, \ldots\} \in \mathcal{I}$ (countable additivity).

The following collections form important ideals in a topological space $(X, \tau)$: the ideal of all finite subsets $\mathcal{F}$, the ideal of all countable subsets $\mathcal{C}$, the ideal of all closed and discrete sets $\mathcal{CD}$, the ideal of all nowhere dense sets $\mathcal{N}$, the ideal of all meager sets $\mathcal{M}$, the ideal of all scattered sets $\mathcal{S}$ (here $X$ must be $T_D$ [12]) and the ideal of all Lebesgue null sets $\mathcal{L}$.

By $(X, \tau, \mathcal{I})$ we will denote a topological space $(X, \tau)$ and an ideal $\mathcal{I}$ on $X$ with no separation properties assumed on $X$. For a space $(X, \tau, \mathcal{I})$ and a subset $A \subseteq X$, $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ is called the *local function* of $A$ with respect to $\mathcal{I}$ and $\tau$ [34]. We simply write $A^*$ instead of $A^*(\mathcal{I})$ in case there is no chance for confusion.

Note that $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ (also denoted by $\tau^*$ when there is no chance for confusion), finer than $\tau$.

The topology $\tau$ of a space $(X, \tau, \mathcal{I})$ is *compatible with the ideal* $\mathcal{I}$ [38], denoted $\tau \sim \mathcal{I}$, if the following condition holds for every subset $A$ of $X$: if for every $x \in A$ there exists a $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. An ideal $\mathcal{I}$ in a topological space $(X, \tau, \mathcal{I})$ is called *local relative to the topology* [42] or has the strong localization property if any subset of $X$ which is locally in $\mathcal{I}$ is in $\mathcal{I}$ (a set $A$ is locally in $\mathcal{I}$ [42] if $A \cap A^*(\mathcal{I}) = \emptyset$). For example, the $\sigma$-ideal of meager (= first category) sets is always local whereas every topology is compatible with the ideal of meager subsets – this result is known as the Banach category theorem. Clearly an ideal $\mathcal{I}$ on a space $(X, \tau, \mathcal{I})$ is local if and only if it is compatible with the topology $\tau$.

Given a space $(X, \tau, \mathcal{I})$ and $A \subseteq X$, $A$ is called *$\mathcal{I}$-open* [29] if $A \subseteq \mathrm{Int}(A^*)$. A space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-Hausdorff* [11] if for each two distinct points $x \neq y$, there exist $\mathcal{I}$-open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $U \cap V = \emptyset$.

For more results on topological ideals, besides the ones from the references given above, the reader may refer (for example) to [20, 21, 23, 30].

# 2 $\mathcal{I}$-dense sets and $\mathcal{I}$-resolvable spaces

**Definition 1** A subset $A$ of a topological space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-dense* if every point of $X$ is in the local function of $A$ with respect to $\mathcal{I}$ and $\tau$, i.e. if $A^*(\mathcal{I}) = X$.

Clearly every $\mathcal{I}$-dense set is $\tau^*$-dense and hence dense. Further, $X$ need not always be $\mathcal{I}$-dense.

**Example 2.1** A $\tau^*$-dense set need not always be $\mathcal{I}$-dense. Let $X = \{a, b, c, \}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Set $A = \{a, b\}$. It is easily seen that $A$ is $\tau^*$-dense and that $A^*(\mathcal{I}) = \{b, c\}$.

An ideal $\mathcal{I}$ is *codense* if each of its members is codense. Note that an ideal $\mathcal{I}$ is codense if and only if $\tau \cap \mathcal{I} = \{\emptyset\}$.

**Theorem 2.2** *For a nonempty topological space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) *Every nonempty open set is $\mathcal{I}$-dense, i.e. $X$ is $\mathcal{I}$-hyperconnected.*

(2) *$(X, \tau)$ is hyperconnected and $\mathcal{I}$ is codense.*

*Proof.* (1) $\Rightarrow$ (2) Clearly every $\mathcal{I}$-hyperconnected space is hyperconnected. Let $U$ be open, nonempty and a member of the ideal. By (1), $U^*(\mathcal{I}) = X$. On the other hand, since $U \in \mathcal{I}$, $U^*(\mathcal{I}) = \emptyset$. Hence $X = \emptyset$. By contradiction, $\mathcal{I}$ is codense.

(2) $\Rightarrow$ (1) Let $\emptyset \neq U \in \tau$. Let $x \in X$. Due to the hyperconnectedness of $(X, \tau)$, every open neighborhood $V$ of $x$ meets $U$. Moreover, $U \cap V$ is an open non-ideal set, since $\mathcal{I}$ is codense. Thus $x \in U^*(\mathcal{I})$. This shows that $U$ is $\mathcal{I}$-dense. $\square$

**Definition 2** A nonempty topological space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-resolvable* if $X$ has two disjoint $\mathcal{I}$-dense subsets.

**Remark 2.3** Note that it is equivalent to stipulate that the resolving $\mathcal{I}$-dense sets be disjoint modulo $\mathcal{I}$, i.e. their intersection is an element of the ideal.

However, every resolvable space is $\mathcal{N}$-resolvable and generally, if $\mathcal{I}$ and $\mathcal{J}$ are ideals with $\mathcal{I}$ contained in $\mathcal{J}$, $X$ is $\mathcal{J}$-resolvable implies that $X$ is $\mathcal{I}$-resolvable. Thus we have the following result:

**Theorem 2.4** *For a nonempty topological space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) $(X, \tau)$ *is resolvable.*

(2) $(X, \tau)$ *is $\mathcal{N}$-resolvable.*

(3) $(X, \tau)$ *is $\{\emptyset\}$-resolvable.* $\square$

In Section 4, the concept of a completely codense ideal is introduced. For now we just note that completely codense ideals are precisely those whose members are nowhere dense. As a consequence we have that resolvability implies $\mathcal{I}$-resolvability granted $\mathcal{I}$ is completely codense.

The maximum ideal $\mathcal{P}(X)$ is an obstruction to $\mathcal{I}$-resolvability, i.e. every nonempty topological space is $\mathcal{P}(X)$-irresolvable; moreover every space is $\mathcal{P}(X)^-$-irresolvable, where $\mathcal{P}(X)^-$ is the ideal formed by excluding a given singleton from the maximal ideal. Below it is also noted that $X$ is $\mathcal{I}$-irresolvable if $\mathcal{I}$ contains any nonempty open set.

**Questions.** Do any other proper ideals also prevent resolvability? Can the strength of classical resolvability be measured by the 'size' of the obstructing ideals?

For a cardinal $\kappa$, a space $(X, \tau)$ is *$\kappa$-resolvable* if there is a family of $\kappa$-many pairwise disjoint dense subsets of $(X, \tau)$. According to this terminology "resolvable" coincides with

2-resolvable. Ceder [5] has shown that a Hausdorff space $(X, \tau)$ is $\Delta(X)$-resolvable provided that $\aleph_0 \leq \omega(X) \leq \Delta(X)$, where $\omega(X)$ denotes the weight of $(X, \tau)$ and $\Delta(X)$ the dispersion character [26] of $(X, \tau)$, i.e. $\Delta(X) = \min\{|U| : U \neq \emptyset$ is open in $(X, \tau)\}$.

Whenever $(X, \tau)$ is $\Delta(X)$-resolvable with $\aleph_0 \leq \Delta(X)$, one can not always find disjoint dense subsets $\{D_\alpha : \alpha < \Delta(X)\}$ such that for each $\alpha$ and each nonempty open set $U$ we have $|U \cap D_\alpha| = \Delta(X)$ as a topological sum of $2^{2^{\aleph_0}}$ copies of the reals shows. However, the claim is true if $(X, \tau)$ is the Real line and this follows from the following argument: We know that the dispersion character of the reals is $\mathbf{c}$, and we have $\mathbf{c}.\mathbf{c} = \mathbf{c}$. So pick $\mathbf{c}.\mathbf{c}$ disjoint dense sets $E(i, j)$, $i, j < \mathbf{c}$. Build $\mathbf{c}$ (disjoint) dense sets $D(i)$ by setting $D(i)$ to be the union of the $E(i, j)$ (with index $j$). Hence every nonempty open set intersects each $D(i)$ in $\mathbf{c}$ points. Now, by the result of Ceder it is obvious that the usual space of reals is resolvable with respect to the ideal of sets of cardinality less than $\Delta$. Moreover, since $|U \cap D_\alpha|$ has cardinality equal to the dispersion character of the usual reals and since this dispersion character equals $2^{\aleph_0}$, under the negation of the continuum hypothesis, $\aleph_1 <$ the dispersion character and so, the space of reals would be $\mathcal{I}$-resolvable where $\mathcal{I}$ is the ideal of subsets of cardinality at most $\aleph_1$ or the ideal of sets of cardinality strictly less than $\aleph_2$.

**Questions.** Is there a space which is $\aleph_0$-resolvable but not $\Delta$-resolvable? What if the space satisfies strong separation axioms? What about the analogues of these two questions for resolvability modulo an ideal?

More generally, can resolvability modulo an ideal shed some light on spaces which are exactly $\alpha$-resolvable for some cardinal $\alpha$? What if the space is at least $\alpha$-resolvable? At most $\alpha$-resolvable for some $\alpha < \Delta(X)$?

Codense ideals are called $\tau$-boundary ideals in [37] where the following is noted.

**Theorem 2.5** *An ideal of subsets of a space $X$ is codense if and only if $X = X^*$.*

*Proof.* Sufficiency: If $U$ is a nonempty open subset and $X = X^*$, then $U = U \cap X$ is not an element of the ideal $\mathcal{I}$. So, $\mathcal{I}$ is codense. Necessity: If $\mathcal{I}$ is codense, then $X \setminus X^*$ must be empty, since otherwise it would contain a nonempty open set $U$, whose intersection with $X$ belongs to $\mathcal{I}$. $\square$

Evidently, if $\mathcal{I}$ is not codense, no subset of $X$ is $\mathcal{I}$-dense, not even $X$ itself.

**Theorem 2.6** *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable, then $\mathcal{I}$ is codense.*

*Proof.* Let $A$ be a subset of $(X, \tau, \mathcal{I})$ such that $A$ forms an $\mathcal{I}$-resolution with its complement. From [28, Theorem 2.3 (a)], it follows that $X$ is $\mathcal{I}$-dense. Thus by Theorem 2.5, $\tau \cap \mathcal{I} = \{\emptyset\}$. $\Box$

**Corollary 2.7** *If $(X, \tau)$ is $\mathcal{M}$-resolvable, then $X$ is a Baire space.* $\Box$

For the converse, note that a Baire space need not be even resolvable. However, the production of spaces which are both Baire and irresolvable (and have some reasonable separation properties) seems to demand additional set theoretic assumptions beyond ZFC (see [33]).

**Example 2.8** Let $X$ be the usual real numbers. Let $\mathcal{I}$ be the power set of the set $\mathbf{Q}$ of rationals and let $\mathcal{J}$ be the power set of the set $\mathbf{P}$ of irrationals. Then, $X$ is both $\mathcal{I}$-resolvable and $\mathcal{J}$-resolvable. But, $X$ is not $(\mathcal{I} \vee \mathcal{J})$-resolvable, since $(\mathcal{I} \vee \mathcal{J})$ is the power set of $X$.

**Question.** If $X$ is $\mathcal{I}$-resolvable, then is $X$ $(\mathcal{I} \vee \mathcal{N})$-resolvable? It is known that $\mathcal{I} \vee \mathcal{N}$ is codense if $\mathcal{I}$ is codense.

A yes answer would tell us to restrict our attention to codense ideals containing $\mathcal{N}$.

Given a topological space $(X, \tau)$, the collection of all regular open sets forms a base for a topology $\tau_s$, coarser than $\tau$, called *the semi-regularization.* The topology $\tau$ is called *s-equivalent* [40] to a topology $\sigma$ on $X$ if $\tau$ and $\sigma$ have same semi-regularizations.

**Theorem 2.9** *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable, then $\tau$ and $\tau^*$ are s-equivalent.* $\Box$

**Theorem 2.10** *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable and the scattered sets of $(X, \tau^*)$ are in $\mathcal{I}$, then $\tau \sim \mathcal{I}$ and $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Hausdorff.*

*Proof.* The compatibility between $\tau$ and $\mathcal{I}$ as well as the fact that singletons are members of the ideal follows from [28, Theorem 5.4]. Let next $A$ and $B$ be disjoint $\mathcal{I}$-dense subsets of $X$ such that $X = A \cup B$. Note that both $A$ and $B$ are $\mathcal{I}$-open. Let $x, y \in X$. In order to show that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-Hausdorff, we need to consider only the case when both $x$ and $y$ are (for example) in $A$. It is easily observed (see [28, Theorem 2.3 (h)]) that $U = A \setminus \{y\}$ and $V = B \cup \{y\}$ are disjoint $\mathcal{I}$-open sets containing $x$ and $y$ respectively. $\square$

A point $x$ of a space $(X, \tau, \mathcal{I})$ is called *inexhaustibly approached* by a set $A$ if $x \in A^*(\mathcal{I})$. Clearly the set of all inexhaustibly approached points by a set $A$ is precisely the local function of $A$. In [3], Blumberg introduced the definition for $\mathcal{I} = \mathcal{M}$.

**Theorem 2.11** *If $(X, \tau)$ is $\mathcal{M}$-resolvable (resp. $\mathcal{N}$-resolvable) and no point of $X$ is inexhaustibly approached by itself, then $X$ is $\mathcal{M}$-Hausdorff (resp. $\mathcal{N}$-Hausdorff).*

*Proof.* From the Banach Category Theorem and [28, Theorem 4.11] it follows that both $\mathcal{M}$ and $\mathcal{N}$ are local ideals. Thus from [28, Theorem 4.5], we have that each point of $X$ is in $\mathcal{M}$ (resp. in $\mathcal{N}$), since none of them is inexhaustibly approached by itself. Now proceeding as in the proof of Theorem 2.10, we can easily conclude that $X$ is $\mathcal{M}$-Hausdorff (resp. $\mathcal{N}$-Hausdorff). $\square$

# 3 The resolvability of the density topology

**Definition 3** [24, 46] A measurable set $E \subseteq \mathbf{R}$ has density $d$ at $x \in \mathbf{R}$ if

$$\lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

exists and is equal to $d$. Set $\phi(E) = \{x \in \mathbf{R} \colon d(x, E) = 1\}$. Let $A \sim B$ mean that the symmetric difference of $A$ and $B$ has measure zero, i.e. $A \triangle B$ is a nullset.

The open sets of the density topology $\tau_d$ are those measurable sets $E$ that satisfy $E \subseteq \phi(E)$. Clearly the density topology $\tau_d$ is finer than the usual topology on the real line.

The following theorem gives a positive answer to the question from [13].

**Theorem 3.1** *The density topology is resolvable.*

*Proof.* According to item 12 of Chapter 8 from [17] (also [19, page 70]), there exists a set $D$ such that for every measurable set $A$, $m_*(D \cap A) = \sup\{m(B) | B \subseteq D \cap A$ and $B$ is measurable$\} = 0$ and $m^*(D \cap A) = \inf\{m(B) | D \cap A \subseteq B$ and $B$ is measurable$\} = m(A)$. Neither $D$ nor its complement are measurable. If $U$ is any nonempty element of the density topology, $U$ is measurable with $m(U) > 0$. This shows that $U$ contains points in the complement of $D$ for otherwise, $m_*(D \cap U) \geq m(U) > 0$. Also, $U \cap D \neq \emptyset$, since $m^*(D \cap U) = m(U) > 0$ whereas, $m^*(\emptyset) = 0$. Clearly, $D$ and its complement form a resolution for the set of reals with the density topology. $\square$

Moreover, every dense set $D$ which forms a resolution with its complement contains no set of positive measure. For sets of positive measure are precisely the $\tau_d$-somewhere dense sets which have the $\tau_d$-property of Baire [43]. In particular, if $m(E) > 0$, there exists $U \in \tau_d$, the density topology, such that $U \triangle E$ is $\tau_d$-nowhere dense (i.e. $U \sim E$). Evidently, $V = U \setminus \mathrm{Cl}_{\tau_d}(U \setminus E) \in \tau_d$ and since $m(V) = m(U) = m(E) > 0$, $V \neq 0$. Hence, $E \subseteq D \Rightarrow D$ is not codense. This implies also that such a $D$ cannot have a defined density at any point. For if $D$ had a positive density at some point $x$, $D$ would contain a subset $E = [x-h, x+h] \cap D$ of positive measure. On the other hand, if $D$ had density 0 at some point $x$, then $D'$, the complement of $D$, would have a positive density at $x$ resulting in the conclusion that $D'$ is not codense.

**Corollary 3.2** *The density topology $\tau_d$ is $\mathcal{M}$-resolvable.*

*Proof.* Theorem 2.4 and Theorem 3.1 imply that the density topology is $\mathcal{N}$-resolvable. Since in the density topology $\mathcal{M} = \mathcal{N}$ [46], then $\tau_d$ is $\mathcal{M}$-resolvable. $\square$

**Question.** Is the density topology $\aleph_0$-resolvable? If so, is it $\Delta(X)$-resolvable?

Let $D$ be a dense subset of a topological space $(X, \tau)$. Then it is easily checked that $N \subseteq D$ is nowhere dense in $D$ if and only if $N$ is nowhere dense in $(X, \tau)$. Therefore we have:

**Theorem 3.3** *A space $(X, \tau)$ is $\mathcal{M}$-resolvable if and only if $(X, \tau)$ has two disjoint dense Baire subspaces.* $\square$

As a consequence of the theorem above we have that every $\mathcal{M}$-resolvable space is a Baire space (Corollary 2.7).

Let $X$ be the usual space of reals, and $B$ denote a Bernstein set in $X$, i.e. both $B$ and $X \setminus B$ intersect every uncountable closed subset. It is known that a Bernstein set is a Baire subspace (and clearly also dense). Since the complement of a Bernstein set is also a Bernstein set we have by Theorem 3.3 that:

**Theorem 3.4** *The usual space of reals is $\mathcal{M}$-resolvable.* $\square$

**Remark 3.5** That a Bernstein set is a Baire subspace follows from the fact that each uncountable $G_\delta$-set in $X = \mathbf{R}$ contains an uncountable, closed, nowhere dense subset.

**Theorem 3.6** *For a space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) $(X, \tau)$ *is $\mathcal{I}$-resolvable.*

(2) $(X, \tau^*)$ *is resolvable.*

*Proof.* (1) $\Rightarrow$ (2) Let $A$ and $B$ be the sets that form the ideal resolution of $X$. Note that $\mathrm{Cl}^*(A) = A \cup A^* = A \cup X = X$. Hence $A$ and $B$ are $\tau^*$-dense. Thus $(X, \tau^*)$ is resolvable.

(2) $\Rightarrow$ (1) Assume now that $X = A \cup B$, $A \cap B = \emptyset$ and both $A$ and $B$ are $\tau^*$-dense. Let $x \in X$. If $x \notin A^*$, then for some $\tau$-open set $U$ containing $x$, we have $V = U \cap A \in \mathcal{I}$. Note that $V$ is nonempty and moreover $U \not\subseteq A$, since otherwise $B$ fails to be $\tau^*$-dense. Clearly $\emptyset \neq W = U \setminus V \in \tau^*$ and $W \cap A = \emptyset$. Our construction of a $\tau^*$-open nonempty set which is disjoint from $A$ contradicts with the initial assumption. Thus $x \in A^*$ and hence $A$ is $\mathcal{I}$-dense. A similar argument shows that $B$ is $\mathcal{I}$-dense. Thus $(X, \tau)$ is $\mathcal{I}$-resolvable. $\square$

Recall that $A \subseteq (X, \tau, \mathcal{I})$ is called $\star$-*dense-in-itself* [25] iff $A \subseteq A^*$.

**Corollary 3.7** *If $X$ is $\mathcal{I}$-resolvable then there exist disjoint $\mathcal{I}$-dense sets $A$ and $B$ each $\star$-dense-in-itself, i.e., $X$ has an $\mathcal{I}$-resolution of sets which are each $\star$-dense-in-itself.* $\square$

The resolvability of the density topology can be used to prove that the reals are resolvable relative to the $\sigma$-ideal of countable sets.

**Lemma 3.8** *If $\tau$ and $\tau^*$ are any topologies with $\tau$ contained in $\tau^*$, then $(X, \tau^*)$ is resolvable implies that $(X, \tau)$ is resolvable.*

**Theorem 3.9** *Let $\mathcal{C}$ be the $\sigma$-ideal of countable subsets of the reals $X$ with usual topology $\tau$. Let $\mathcal{L}$ be the $\sigma$-ideal of Lebesgue null sets and let $\tau_d$ be the density topology on $X$. Then $(X, \tau)$ is $\mathcal{L}$-resolvable and is therefore also $\mathcal{C}$-resolvable.*

*Proof.* Since $\mathcal{C}$ is contained in $\mathcal{L}$ and null sets are closed in $\tau_d$, the result follows from the inclusion of $\tau^*(\mathcal{C}) \subseteq \tau^*(\mathcal{L})$ and the inclusion of $\tau^*(\mathcal{L}) \subseteq \tau_d$. $\Box$

The topology $\tau^*(\mathcal{L})$ above was studied in 1971 by Scheinberg [44]. In that paper he also considers a 'maximal' extension of the density topology which he calls $U$.

**Question.** Is $(X, U)$ resolvable?

# 4 Maximal $\mathcal{I}$-resolvability

A topological space $(X, \tau, \mathcal{I})$ is called *maximal $\mathcal{I}$-resolvable* if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable and $(X, \sigma, \mathcal{I})$ is not $\mathcal{I}$-resolvable for every topology $\sigma$ which is strictly finer than $\tau$. Note that $\{\emptyset\}$-resolvable spaces are called maximal resolvable. However, often in the literature the term maximally resolvable space is often used to refer to a space $X$ which is $\Delta(X)$-resolvable.

A subset $S$ of a space $(X, \tau, \mathcal{I})$ is a topological space with an ideal $\mathcal{I}_S = \{I \in \mathcal{I} \colon I \subseteq S\} = \{I \cap S \colon I \in \mathcal{I}\}$ on $S$ [11].

**Theorem 4.1** *Nonempty $\tau^*$-open subspaces of $\mathcal{I}$-resolvable spaces are $\mathcal{I}$-resolvable.*

*Proof.* First note that it is an easy exercise to show that for each $U \in \tau^*$, $\tau^*|U = (\tau|U)^*$. The result now follows instantly from Theorem 3.6, since resolvability is open hereditary. That is, if $(X, \tau)$ is $\mathcal{I}$-resolvable and $U$ is $\tau^*$-open, then $(X, \tau^*)$ is resolvable so that $(U, \tau^*|U) = (U, (\tau|U)^*)$ is resolvable and thus, $(U, \tau|U)$ is $\mathcal{I}$-resolvable. $\Box$

**Corollary 4.2** *$\mathcal{I}$-resolvability is open hereditary.* $\Box$

**Theorem 4.3** *Simple expansions of $\mathcal{I}$-resolvable topologies over $\mathcal{I}$-resolvable subspaces are $\mathcal{I}$-resolvable.*

*Proof.* Let $(X, \tau, \mathcal{I})$ be $\mathcal{I}$-resolvable and $S \subseteq X$ be $\mathcal{I}$-resolvable (as a subspace). Let $(D, D')$ be the $\mathcal{I}$-resolution of $(S, \tau_S, \mathcal{I}_S)$. We consider the following two cases:

*Case 1.* $S$ is $\tau^*$-dense in $(X, \tau, \mathcal{I})$, i.e. $S \cup S^* = X$. We prove first that $D$ is $\mathcal{I}$-dense in $(X, \tau, \mathcal{I})$. Let $x \in X$. Assume that for some $U \in \tau$ with $x \in U$ we have $U \cap D \in \mathcal{I}$. We have the following two subcases:

*Subcase 1.* $x \in S$. Then $V = U \cap S \in \tau_S$ is an open neighborhood of $x$ in $(S, \tau_S, \mathcal{I}_S)$ such that $V \cap D = U \cap S \cap D \in \mathcal{I}$ due to the heredity of $\mathcal{I}$. This contradicts the fact that $D$ is $\mathcal{I}$-dense in $(S, \tau_S, \mathcal{I}_S)$. So $D$ is $\mathcal{I}$-dense in $(X, \tau, \mathcal{I})$.

*Subcase 2.* $x \notin S$. Since $X = S \cup S^*$, then $x \in S^*$. In order to prove that $x \in D^*(\mathcal{I})$, we assume the contrary, i.e. there exists $U \in \tau$ with $x \in U$ such that $U \cap D \in \mathcal{I}$. Note that $U \cap S \neq \emptyset$ (otherwise $x \notin S^*$). Pick $y \in U \cap S \in \tau_S$. Since $U \cap D \in \mathcal{I}$, then by heredity of $\mathcal{I}$, $U \cap S \cap D \in \mathcal{I}$. Hence $D$ is not $\mathcal{I}$-dense in $(S, \tau_S, \mathcal{I}_S)$. By contradiction $x \in D^*(\mathcal{I})$, i.e. $D$ is $\mathcal{I}$-dense in $(X, \tau, \mathcal{I})$.

We have thus shown that $D^*(\mathcal{I}) = X$. By a similar argument $D'^*(\mathcal{I}) = X$.

Now let $x \in X$ and let $U \cup (V \cap S)$ be an open neighborhood of $x$ in $(X, \tau(S), \mathcal{I})$, where $\tau(S)$ is the simple expansion of $\tau$ over $S$. If $(U \cup (V \cap S)) \cap D \in \mathcal{I}$, then by heredity of $\mathcal{I}$, $(V \cap S) \cap D$ is a member of $\mathcal{I}$ so that $V$ is empty. Of course, $(V \cap S) \cap D$ cannot be a member of $\mathcal{I}$ if $V$ is nonempty since then $V$ must contain an element of $S$. Hence, $x$ belongs to $U \cap D$ which also cannot be a member of $\mathcal{I}$ since $D^*(\mathcal{I}) = X$. This contradiction shows that $D$ is $\tau(S)^*$-dense. With a similar argument for $D'$ we conclude that $(X, \tau(S), \mathcal{I})$ is $\mathcal{I}$-resolvable.

*Case 2.* $S$ is not $\tau^*$-dense in $(X, \tau, \mathcal{I})$. Then $S' = X \setminus \mathrm{Cl}^*(S)$ is $\tau^*$-open and nonempty. By Theorem 4.1, $S'$ is $\mathcal{I}$-resolvable (more precisely said $\mathcal{I}_S$-resolvable). Let $(E, E')$ be the $\mathcal{I}$-resolution of $S'$. By similar arguments to the ones of Case 1, we can prove that $(D \cup E, D \cup E')$ is $\mathcal{I}$-resolution of $(X, \tau, \mathcal{I})$. Furthermore, using the same technique as at the end of Case 1, we see that $(X, \tau(S), \mathcal{I})$ is $\mathcal{I}$-resolvable. $\square$

By $IR(X)$ we denote the collection of all $\mathcal{I}$-resolvable subspaces of a given space $(X, \tau, \mathcal{I})$.

**Theorem 4.4** *For a topological space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is maximally $\mathcal{I}$-resolvable.*

(2) *$\tau \setminus \{\emptyset\} = IR(X)$.*

*Proof.* (1) $\Rightarrow$ (2) Open nonempty subspaces of $\mathcal{I}$-resolvable spaces are $\mathcal{I}$-resolvable (Corollary 4.2). Let next $S \in IR(X)$. By Theorem 4.3 above $\tau(S)$ is an $\mathcal{I}$-resolvable topology on $X$ finer than $\tau$. By (1), $S \in \tau(S) = \tau$.

(2) $\Rightarrow$ (1) Since $X \in \tau$, then by (2), $X$ is $\mathcal{I}$-resolvable. Assume that $X$ is not maximally $\mathcal{I}$-resolvable and let $\sigma$ be $\mathcal{I}$-resolvable topology strictly finer than $\tau$. Let $U \in \sigma \setminus \tau$. Clearly $U$ is $\mathcal{I}$-resolvable in $(X, \sigma, \mathcal{I})$ and hence in $(X, \tau, \mathcal{I})$. By (2), $U \in \tau$. By contradiction (1) is proved. $\square$

**Corollary 4.5** [6] (i) *Simple expansions of resolvable topologies over resolvable subspaces are resolvable.*

(ii) *A space is maximal resolvable if and only if the collections of all open nonempty and all resolvable subspaces coincide.*

*Proof.* Set $\mathcal{I} = \{\emptyset\}$ in Theorem 4.3 and Theorem 4.4. $\square$

According to El'kin [14] a topological space $(X, \tau)$ is *globally disconnected* if every set which can be placed between an open set and its closure is open, i.e. if every semi-open set is open. Note that the density topology is not globally disconnected, it is not even extremally disconnected. Hence there exists a semi-open set $S$ in the density topology $\tau_d$ such that $S \not\in \tau_d$. Clearly $S$ is resolvable as a subspace and so in the notion of the above given characterization of maximal resolvability we have the following result:

**Corollary 4.6** *The density topology is not maximal resolvable.* $\square$

Our next result follows from the remarks about semi-open subspaces of resolvable spaces and the reason explaining why the density topology is not maximal resolvable and of course, the theorem on simple expansions of resolvable spaces by a resolvable subspace.

**Theorem 4.7** *If a topological space $(X, \tau)$ is maximal resolvable, then $(X, \tau)$ is globally disconnected.* $\square$

**Questions.** Which globally disconnected spaces (if any) are resolvable? Is a resolvable globally disconnected space maximally resolvable? Is there a method for constructing maximal resolvable spaces?

Recall that the ideal defined on the topological sum $X = \sum_{\alpha\in\Omega} X_\alpha$ of the family of spaces $(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha)_{\alpha\in\Omega}$ is $\mathcal{I} = \bigvee_{\alpha\in\Omega} \mathcal{I}_\alpha = \{\cup_{\alpha\in\Omega} I_\alpha \colon I_\alpha \in \mathcal{I}_\alpha\}$ [11].

**Theorem 4.8** *Let $(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha)_{\alpha\in\Omega}$ be a family of topological spaces. For the topological sum $X = \sum_{\alpha\in\Omega} X_\alpha$ the following conditions are equivalent:*

(1) *$X$ is a $\mathcal{I}$-resolvable.*

(2) *Each $X_\alpha$ is $\mathcal{I}_\alpha$-resolvable.*

*Proof.* (1) $\Rightarrow$ (2) follows from Theorem 4.1.

(2) $\Rightarrow$ (1) Let $(A_\alpha, B_\alpha)$ be the $\mathcal{I}_\alpha$-resolution of each $X_\alpha$. Set $A = \cup_{\alpha\in\Omega} A_\alpha$ and $B = \cup_{\alpha\in\Omega} B_\alpha$. We claim that $A$ and $B$ form the $\mathcal{I}$-resolution of $X$. In order to show first that $A$ is $\mathcal{I}$-dense in $X$, choose a point $x \in X$. If $x$ is not in the local function of $A$, then there exists an open set $U$ of $X$ containing $x$ such that $U \cap A \in \mathcal{I}$. Let $\alpha$ be the index for which $x \in X_\alpha$. Due to the heredity of $\mathcal{I}$, $W = (U \cap X_\alpha) \cap (A \cap X_\alpha) \in \mathcal{I}$. Note that $W \in \mathcal{I}_\alpha$. Moreover, $U \cap X_\alpha$ is a $\tau_\alpha$-open neighborhood of $x$ meeting $A_\alpha$ in an element of $\mathcal{I}_\alpha$, which shows that $(X_\alpha, \tau_\alpha, \mathcal{I}_\alpha)$ is not $\mathcal{I}_\alpha$-resolvable. By contradiction $A$ is a $\mathcal{I}$-dense. In a similar way, one shows that $B$ is $\mathcal{I}$-dense. Hence $X$ is a $\mathcal{I}$-resolvable. $\square$

Given an ideal $\mathcal{I}$ and a resolvable topological space $(X, \tau)$, we have the natural question: When is $X$ $\mathcal{I}$-resolvable? Recall that a set $A$ is called *locally dense* or *preopen* if $A \subseteq \mathrm{Int}\overline{A}$. It is shown in [16] that $A$ is preopen if $A = U \cap D$, where $U$ is open and $D$ is dense. The collection of all preopen subsets of a space $(X, \tau)$ will be denoted (as usual) by $PO(X)$. We call an ideal $\mathcal{I}$ on a space $(X, \tau, \mathcal{I})$ a *completely codense* if $PO(X) \cap \mathcal{I} = \{\emptyset\}$. Note that if $(\mathbf{R}, \tau)$ is the real line with the usual topology, then $\mathcal{C}$ is codense but not completely codense.

**Theorem 4.9** *If $(X, \tau, \mathcal{I})$ is a partition space, then $\mathcal{I}$ is completely codense if and only if $\mathcal{I}$ is the minimal ideal.* $\square$

**Theorem 4.10** *For a topological space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*

(1) *$\mathcal{I}$ is a completely codense.*

(2) *Every $\tau$-dense set is $\mathcal{I}$-dense.*

*Proof.* (1) $\Rightarrow$ (2) Let $D \subseteq X$ be dense in $(X, \tau)$. Let $x \in X$ and let $U$ be an open neighborhood of $x$. Clearly $A = U \cap D \neq \emptyset$ and $A \in PO(X)$. Hence, by (1), $A \notin \mathcal{I}$. Thus $x \in D^*$. So, $D$ is $\mathcal{I}$-dense.

(2) $\Rightarrow$ (1) Let $A \in PO(X)$ such that $A \neq \emptyset$ and $A \in \mathcal{I}$. Then $A = U \cap D$, where $U \in \tau$ and $D$ is $\tau$-dense. Let $x \in A$. Now, $U$ is an open neighborhood of $x$ such that $U \cap D \in \mathcal{I}$. Thus $x \notin D^*$, hence $D^* \neq X$. So, $D$ is a $\tau$-dense set that fails to be $\mathcal{I}$-dense. By contradiction $PO(X) \cap \mathcal{I} = \{\emptyset\}$. $\square$

**Corollary 4.11** *If $\mathcal{I}$ is completely codense, then $(X, \tau)$ is resolvable if and only if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable.* $\square$

**Remark 4.12** The requirement that the ideal is a completely codense is necessary. One can easily find a resolvable space $(X, \tau, \mathcal{I})$, where $\mathcal{I}$ is codense but $(X, \tau, \mathcal{I})$ fails to be $\mathcal{I}$-resolvable.

**Theorem 4.13** *An ideal $\mathcal{I}$ is completely codense on $(X, \tau)$ if and only if $\mathcal{I} \subseteq \mathcal{N}$, i.e. if each member of $\mathcal{I}$ is nowhere dense.*

*Proof.* Let $A \in \mathcal{I}$. Then $A \cap \mathrm{Int}\overline{A} \in PO(X)$ and so $A \cap \mathrm{Int}\overline{A} = \emptyset$. Hence $\mathrm{Int}\overline{A} = \emptyset$, i.e. $A \in \mathcal{N}$. Conversely, let $A \in PO(X)$ and $A \in \mathcal{I}$. Since $A \subseteq \mathrm{Int}\overline{A}$ and $\mathrm{Int}\overline{A} = \emptyset$, we have $A = \emptyset$. Hence $\mathcal{I}$ is completely codense. $\square$

**Theorem 4.14** *A topological space $X$ is maximal resolvable if and only if $X$ is maximal $\mathcal{I}$-resolvable for each completely codense ideal $\mathcal{I}$.* $\square$

*Proof.* One direction of the theorem is easy. If $(X, \tau)$ is maximally $\mathcal{I}$-resolvable and $\mathcal{J}$-resolvable where $\mathcal{I}$ is contained in $\mathcal{J}$, then $(X, \tau)$ is maximally $\mathcal{J}$-resolvable. Thus, if $\mathcal{I}$

is contained in $\mathcal{N}$ and $(X,\tau)$ is maximally resolvable, then certainly, $(X,\tau)$ is $\mathcal{N}$-resolvable and hence $\mathcal{I}$-resolvable so that $(X,\tau)$ is maximally $\mathcal{I}$-resolvable.

For the converse, suppose that $(X,\tau)$ is maximally $\mathcal{I}$-resolvable with $\mathcal{I}$ contained in $\mathcal{N}$. Then by the above, $(X,\tau)$ is maximally $\mathcal{N}$-resolvable. Since each open set in $\tau^*(\mathcal{N}) = \tau^\alpha$ is an $\mathcal{N}$-resolvable subspace (we are being a little sloppy here but the nowhere dense subsets of such a subspace are simply intersections of global nowhere dense sets with the subspace), evidently, $\tau^*(\mathcal{N}(\tau)) = \tau$. In particular, each member of $\mathcal{I}$ is closed in $(X,\tau)$. This fact could have been argued directly by noting that $(X,\tau)$ is resolvable if and only if $(X,\tau^*(\mathcal{N}(\tau)))$ is resolvable and using the fact that $\mathcal{N}(\tau^*(\mathcal{N}(\tau))) = \mathcal{N}(\tau)$ so that if $(X,\tau)$ is resolvable, then $(X,\tau^*(\mathcal{N}(\tau)))$ is $\mathcal{N}(\tau)$-resolvable. Since $\tau$ is contained in $\tau^*(\mathcal{N}(\tau))$, we must have equality if $(X,\tau)$ is maximally $\mathcal{N}(\tau)$-resolvable. Since all that is needed is that $\tau^*(\mathcal{I})$ be contained in (and hence equal to) $\tau$, Theorem 3.6 could be invoked. In any case, to arrive at a contradiction, let $\sigma$ be a proper expansion of $\tau$ such that $(X,\sigma)$ is resolvable and let $(E \cup F)$ be a resolution for $(X,\sigma)$. Since $(X,\sigma)$ is not $\mathcal{I}$-resolvable ($(X,\tau)$ is maximally $\mathcal{I}$-resolvable), there exists a $\sigma$-open set $U$ such that either $(U \cap E)$ or $(U \cap F)$ is a nonempty member of $\mathcal{I}$. Without loss of generality, assume that $(U \cap E)$ is a nonempty member of $\mathcal{I}$. Thus, $V = X \setminus (U \cap E)$ is $\tau$-open, since members of $\mathcal{I}$ are $\tau$-closed. So, $W = (V \cap U)$ is a nonempty $\sigma$-open set and $(W \cap E)$ is empty. $W$ is nonempty since $U \setminus E = (U \cap F)$ is nonempty and is a subset of $V$. But, $(W \cap E)$ is empty contradicts $(E \cup F)$ being a resolution for $(X,\sigma)$. Evidently, $(X,\tau)$ is maximally resolvable. $\Box$

Recall that a topological space $(X,\tau)$ is called *submaximal* if every dense subset of $X$ is open. Recently, submaximal spaces were studied in [2, 13]. Note that every submaximal space is strongly irresolvable but not vice versa, where a topological space $(X,\tau)$ is *strongly irresolvable* [15] if no nonempty open set is resolvable.

**Theorem 4.15** *An ideal $\mathcal{I}$ on a submaximal space $(X,\tau)$ is codense if and only if it is completely codense.* $\Box$

## 5 Bounded resolvability

A subset $A$ of a topological space $(X, \tau)$ is called *bounded* [35] (resp. *L-bounded*, *parabounded* [22]) if it is contained in some finite union of members (resp. countable union of members, locally finite open refinement) of every open cover of the whole space $X$. The set $A \subseteq (X, \tau)$ will be called *discretely finite* (= df-set) if for every point $x \in A$, there exists $U \in \tau$ containing $x$ such that $U \cap A$ is finite. Clearly every discrete and every finite set is a df-set but not vice versa.

**Example 5.1** We give an example of a df-set that is neither discrete nor finite. The *digital line* or the so called *Khalimsky line* [31, 32] is the set of all integers $\mathbf{Z}$, equipped with the topology $\mathcal{K}$, generated by $\mathcal{G}_{\mathcal{K}} = \{\{2n-1, 2n, 2n+1\} \colon n \in \mathbf{Z}\}$. Let $A$ be union of all even and all prime integers. Note that $A$ is an infinite, non-discrete, df-set.

We denote the ideals of all bounded, L-bounded and parabounded subsets of a space $(X, \tau)$ by $\mathcal{B}$, $\mathcal{LB}$ and $\mathcal{PB}$, respectively. The ideal of all closed df-sets will be denoted by $\mathcal{CDF}$.

**Theorem 5.2** *Every df-set is parabounded.*

*Proof.* Let $A \in (X, \tau)$ be discretely finite. Let $\mathbf{U} = (U_i)_{i \in I}$ be an open cover of $X$. For each $x \in A$ choose an $U_i \in \mathbf{U}$ containing $x$ and an open $U \ni x$ such that $U \cap A \in \mathcal{F}$. Set $U_x = U_i \cap U$. Note that $(U_x)_{x \in A}$ is an open cover of $A$ refining $\mathbf{U}$ and locally finite in $X$. Hence $A$ is parabounded. □

**Corollary 5.3** *For any space $(X, \tau)$, $\mathcal{CDF} \subseteq \mathcal{PB}$.* □

**Remark 5.4** A parabounded set need not be discretely finite. Any infinite set with the point excluded topology is clearly parabounded but not a df-set.

Lambrinos, Reilly and Vamanamurthy [36] define a space $(X, \tau)$ to be *bounded-finite* if every bounded subset of $X$ is finite. Reilly and Vamanamurthy [41] called a topological space $(X, \tau)$ a *cic-space* if every countably infinite set is closed.

**Theorem 5.5** (i) [18] *Let* $(X,\tau)$ *be a* $T_1$*-space. Then* $X$ *is bounded-finite if and only if* $\tau = \tau^*(\mathcal{B})$.

(ii) *Let* $(X,\tau)$ *be a* $T_1$ *cic-space. Then* $X$ *is L-bounded-countable, i.e. every L-bounded subset of* $X$ *is countable if and only if* $\tau = \tau^*(\mathcal{LB})$.

(iii) *Let* $(X,\tau)$ *be a space, where df-sets are closed. Then every para-bounded subset of* $X$ *is a df-set if and only if* $\tau = \tau^*(\mathcal{PB})$.

*Proof.* (i) This is proved in [18].

(ii) Assume first that $X$ is L-bounded-countable and let $U \in \tau^*(\mathcal{LB})$. Note that the family $\beta = \{U \setminus A \colon U \in \tau \text{ and } A \in \mathcal{LB}\}$ is a basis for the topology $\tau^*(\mathcal{LB})$. Since $\mathcal{LB} \subseteq \mathcal{C}$, then each member of $\mathcal{LB}$ is closed, because $X$ is $T_1$ and cic. Thus $\tau^*(\mathcal{LB}) \subseteq \tau$. Hence $\tau = \tau^*(\mathcal{LB})$. Next, let $A$ be L-bounded in $X$. Note that for each $x \in A$, the set $(X \setminus A) \cup \{x\} \in \tau^*(\mathcal{LB})$ and hence is open in $(X,\tau)$. Thus $A$ is discrete subset of $(X,\tau)$ and clearly countable, since $A$ is L-bounded. This shows that $X$ is L-bounded-countable.

(iii) The proof is very similar to the one of (ii), hence we omit it. □

**Corollary 5.6** (i) *A* $T_1$ *bounded-finite space* $(X,\tau)$ *is resolvable if and only if it is* $\mathcal{B}$*-resolvable.*

(ii) *A* $T_1$ *L-bounded-countable cic-space* $(X,\tau)$ *is resolvable if and only if it is* $\mathcal{LB}$*-resolvable.*

(iii) *A space* $(X,\tau)$ *in which parabounded sets are closed is resolvable if and only if it is* $\mathcal{PB}$*-resolvable.*

*Proof.* Follows from Theorem 3.6 and Theorem 5.5. □

A space $(X,\tau)$ is called *B-closed* if every bounded subset of $X$ is closed and *locally bounded* [35] (resp. locally L-bounded, locally parabounded) if every point of $X$ has a bounded (resp. L-bounded, parabounded) neighborhood. The proofs of the following two theorems are left to the reader.

**Theorem 5.7** (i) *A B-closed space* $(X,\tau)$ *is resolvable if and only if it is* $\mathcal{B}$*-resolvable.*

(ii) *Every locally bounded space is* $\mathcal{B}$*-irresolvable.* □

**Theorem 5.8** *Let $(X, \tau)$ be locally bounded (resp. locally L-bounded, locally parabounded) and let $\tau \sim \mathcal{B}$ (resp. $\tau \sim \mathcal{LB}$, $\tau \sim \mathcal{PB}$). Then $\tau \subseteq \mathcal{B}$ (resp. $\tau \subseteq \mathcal{LB}$, $\tau \subseteq \mathcal{PB}$) and hence $X$ is compact (resp. Lindelöf, paracompact).* $\square$

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