Unified operation approach of generalized closed sets via topological ideals*

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Abstract

The aim of this paper is to unify the concepts of topological ideals, operation functions (= expansions) and generalized closed sets.

1 Introduction

The following three different topological concepts were a major point of research in recent years:

- Topological ideals [4, 9, 10, 11, 12, 14, 15, 16, 32, 33].
- Operation function (= expansion) [17, 26, 27, 28, 29, 30, 35, 36, 37].
- Generalized closed sets [1, 2, 3, 5, 6, 8, 21, 22, 23, 24, 25, 34, 37].

An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a non-void collection of subsets of $X$ satisfying the following two properties: (1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ (heredity), and (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (finite additivity). The following collections of sets form important ideals on a space $(X, \tau)$: $\mathcal{F}$ — the ideal of finite subsets of $X$, $\mathcal{C}$ — the ideal of countable subsets of $X$, $\mathcal{CD}$ — the ideal of closed discrete sets in $(X, \tau)$, $\mathcal{N}$ — the ideal of nowhere dense sets in $(X, \tau)$, $\mathcal{M}$ — the ideal of meager sets in $(X, \tau)$, $\mathcal{B}$ — the ideal of all

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bounded sets in \((X, \tau), S\) — the ideal of scattered sets in \((X, \tau)\) (here \(X\) must be \(T_0\)), \(K\) — the ideal of relatively compact sets in \((X, \tau)\).

For a topological space \((X, \tau, \mathcal{I})\) and a subset \(A \subseteq X\), we denote by \(A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}\), written simply as \(A^*\) in case there is no chance for confusion. In [19], \(A^*\) is called the \(local function\) of \(A\) with respect to \(\mathcal{I}\) and \(\tau\). Recall that \(A \subseteq (X, \tau, \mathcal{I})\) is called \(\tau^\ast\)-closed [15] if \(\alpha^\ast\subseteq A\). It is well-known that \(\text{Cl}^\ast(A) = A \cup A^*\) defines a Kuratowski closure operator for a topology \(\tau^\ast(\mathcal{I})\), finer than \(\tau\).

An \(operation\ \gamma\) [17, 26] on the topology \(\tau\) on a given topological space \((X, \tau)\) is a function from the topology itself into the power set \(P(X)\) of \(X\) such that \(\gamma(U) \subseteq V\) for each \(V \in \tau\), where \(V^\gamma\) denotes the value of \(\gamma\) at \(V\). The following operators are examples of the operation \(\gamma\): the closure operator \(\gamma_{cl}\) defined by \(\gamma(U) = \text{Cl}(U)\), the identity operator \(\gamma_{id}\) defined by \(\gamma(U) = U\), the interior-closure operator \(\gamma_{ic}\) defined by \(\gamma(U) = \text{Int}(\text{Cl}(U))\). In [35], the \(\gamma\)-operation is called an \(expansion\). Another example of the operation \(\gamma\) is the \(\gamma_f\)-operator defined by \(\gamma_f(U) = (\text{Fr}U)^c = X \setminus \text{Fr}U\) [35]. Two operators \(\gamma_1\) and \(\gamma_2\) are called \(mutually dual\) [35] if \(U^{\gamma_1} \cap U^{\gamma_2} = U\) for each \(U \in \tau\). For example the identity operator is mutually dual to any other operator, while the \(\gamma_f\)-operator is mutually dual to the closure operator [35].

The following definition contains the concepts of \(generalized closed sets\) used throughout this paper. In Theorem 2.2 of Section 2, it is proved that \(\alpha^{**}g\)-closedness is same as \(g\alpha^{**}\)-closedness.

**Definition 1** A subset \(A\) of a space \((X, \tau)\) is called:

1. a \(generalized closed set\) (briefly \(g\)-closed) [20] if \(\overline{A} \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open,
2. a \(\alpha\)-\(generalized closed set\) (briefly \(\alpha g\)-closed) [22] if \(\alpha\text{Cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open,
3. a \(\alpha^{**}\)-\(generalized closed set\) (briefly \(\alpha^{**}g\)-closed) [22] if \(\alpha\text{Cl}(A) \subseteq \text{IntCl}U\) whenever \(A \subseteq U\) and \(U\) is open,
4. a \(regular generalized closed set\) (briefly \(r\)-\(g\)-closed) [31] if \(\overline{A} \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open,
5. a \(generalized \alpha^{**}\)-\(closed set\) (briefly \(g\alpha^{**}\)-closed) [23] if \(\alpha\text{Cl}(A) \subseteq \text{IntCl}U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open.
2 Basic properties of \((\mathcal{I}, \gamma)\)-generalized closed sets

Definition 2 A subset \(A\) of a topological space \((X, \tau)\) is called \((\mathcal{I}, \gamma)\)-generalized closed if \(A^* \subseteq U^\gamma\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

We denote the family of all \((\mathcal{I}, \gamma)\)-generalized closed subsets of a space \((X, \tau, \mathcal{I}, \gamma)\) by \(IG(X)\) and simply write \(\mathcal{I}\)-generalized closed (= \(\mathcal{I}\)-g-closed) in case when \(\gamma\) is the identity operator.

Theorem 2.1 Every g-closed set is \((\mathcal{I}, \gamma)\)-generalized closed but not vice versa. \(\Box\)

Theorem 2.2 Let \((X, \tau)\) be a topological space and let \(A \subseteq X\). Then:

1. \(A\) is \(\{\emptyset\}\)-g-closed if and only if \(A\) is g-closed.
2. \(A\) is \(\mathcal{N}\)-g-closed if and only if \(A\) is \(\alpha g\)-closed.
3. \(A\) is \((\mathcal{N}, \gamma_{ic})\)-g-closed if and only if \(A\) is \(\alpha^* g\)-closed.
4. \(A\) is \((\{\emptyset\}, \gamma_{ic})\)-g-closed if and only if \(A\) is \(r\)-g-closed.
5. \(A\) is \((\mathcal{N}, \gamma_{ic})\)-g-closed if and only if \(A\) is \(g\alpha^*\)-closed.

Proof. Follow from the facts: \(A^*(\{\emptyset\}) = Cl(A)\) and \(A^*(\mathcal{N}) = Cl(Int(Cl(A))\) and \(A \cup A^*(\mathcal{N}) = \alpha Cl(A)\) [15, Example 2.10]. \(\Box\)

In the notion of Theorem 2.2, majority of the theorems below generalize well-known results related to the classes of generalized closed sets given in Definition 1.

Theorem 2.3 If \(A\) is \(\mathcal{I}\)-g-closed and open, then \(A\) is \(\tau^*\)-closed. \(\Box\)

Lemma 2.4 [13, Theorem II3] Let \((A_i)_{i \in I}\) be a locally finite family of sets in \((X, \tau, \mathcal{I})\). Then \(\bigcup_{i \in I} A_i^*(\mathcal{I}) = (\bigcup_{i \in I} A_i)^*(\mathcal{I})\). \(\Box\)

Theorem 2.5 Let \((X, \tau, \mathcal{I}, \gamma)\) be a topological space.

(i) If \((A_i)_{i \in I}\) is a locally finite family of sets and each \(A_i \in IG(X)\), then \(\bigcup_{i \in I} A_i \in IG(X)\).
(ii) Countable union of \((\mathcal{I}, \gamma)\)-generalized closed sets need not be \((\mathcal{I}, \gamma)\)-generalized closed.
(iii) Finite intersection of \((\mathcal{I}, \gamma)\)-generalized closed sets need not be \((\mathcal{I}, \gamma)\)-generalized closed.
Proof. (i) Let $\bigcup_{i \in I} A_i \subseteq U$, where $U \in \tau$. Since $A_i \in IG(X)$ for each $i \in I$, then $A_i^* \subseteq U^\gamma$. Hence $\bigcup_{i \in I} A_i^* \subseteq U^\gamma$. By Lemma 2.4, $(\bigcup_{i \in I} A_i)^* \subseteq U^\gamma$. Hence $\bigcup_{i \in I} A_i \in IG(X)$.

(ii) In the real line with the usual topology $\{\frac{1}{n}\}$ is $F$-g-closed for each $n \in \omega$, where $\omega$ denotes the set of all positive integers. But the set $A = \bigcup_{n \in \omega} \{\frac{1}{n}\}$ is not $F$-g-closed. Note that $(0,2)$ is an open superset of $A$ but the zero point is in the local function of $A$ with respect to the usual topology and $F$.

(iii) Let $X = \{a,b,c,d,e\}$, $\tau = \{\emptyset, \{a,b\}, \{c\}, \{a,b,c\}, X\}$, $\mathcal{I} = \{\emptyset\}$ and $\gamma = \gamma_{ic}$. Set $A = \{a,c,d\}$ and $B = \{b,c\}$. Clearly, $A, B \in IG(X)$ but $A \cap B = \{c\} \notin IG(X)$. □

Lemma 2.6 If $A$ and $B$ are subsets of $(X, \tau, \mathcal{I})$, then $(A \cap B)^*(\mathcal{I}) \subseteq A^*(\mathcal{I}) \cap B^*(\mathcal{I})$. □

A subset $S$ of a space $(X, \tau, \mathcal{I})$ is a topological space with an ideal $\mathcal{I}_S = \{I \in \mathcal{I}: I \subseteq S\} = \{I \cap S: I \in \mathcal{I}\}$ on $S$ [4].

Lemma 2.7 Let $(X, \tau, \mathcal{I})$ be a topological space and $A \subseteq S \subseteq X$. Then, $A^*(\mathcal{I}_S, \tau|S) = A^*(\mathcal{I}, \tau) \cap S$ holds.

Proof. First we prove the following implication: $A^*(\mathcal{I}_S, \tau|S) \subseteq A^*(\mathcal{I}, \tau) \cap S$.

Let $x \notin A^*(\mathcal{I}, \tau) \cap S$. We consider the following two cases:

Case 1. $x \notin S$: Since $A^*(\mathcal{I}_S, \tau|S) \subseteq S$, then $x \notin A^*(\mathcal{I}_S, \tau|S)$.

Case 2. $x \in S$. In this case $x \notin A^*(\mathcal{I}, \tau)$. There exists a set $V \in \tau$ such that $x \in V$ and $V \cap A \in \mathcal{I}$. Since $x \in S$, we have a set $S \cap V \in \tau|S$ such that $x \in S \cap V$ and $(S \cap V) \cap A \in \mathcal{I}$ and hence $(S \cap V) \cap A \in \mathcal{I}_S$. Consequently, $x \notin A^*(\mathcal{I}_S, \tau|S)$.

Both cases show the implication.

Secondly, we prove the converse implication: $A^*(\mathcal{I}, \tau) \cap S \subseteq A^*(\mathcal{I}_S, \tau|S)$. Let $x \notin A^*(\mathcal{I}, \tau) \cap S$. Then, for some open subset $U \cap S$ of $(S, \tau|S)$ containing $x$, we have $(U \cap S) \cap A \in \mathcal{I}_S$. Since $A \subseteq S$, then $U \cap A \in \mathcal{I}_S \subseteq \mathcal{I}$, i.e., $U \cap A \in \mathcal{I}$ for some $V \in \tau$ containing $x$. This shows that $x \notin A^*(\mathcal{I}, \tau)$. □

Theorem 2.8 Let $(X, \tau, \mathcal{I})$ be a topological space and $A \subseteq S \subseteq X$. If $A$ is $\mathcal{I}_S$-g-closed in $(S, \tau|S, \mathcal{I}_S)$ and $S$ is $\mathcal{I}$-g-closed in $X$, then $A$ is $\mathcal{I}$-g-closed in $X$. 4
Proof. Let $A \subseteq U$ and $U \in \tau$. By assumption and Lemma 2.7, $A^*(\mathcal{I}, \tau) \cap S \subseteq U \cap S$. Then we have $S \subseteq U \cup (X \setminus A^*(\mathcal{I}, \tau))$. Since $X \setminus A^*(\mathcal{I}, \tau) \in \tau$, then $A^*(\mathcal{I}, \tau) \subseteq S^*(\mathcal{I}, \tau) \subseteq U \cup (X \setminus A^*(\mathcal{I}, \tau))$. Therefore, we have that $A^*(\mathcal{I}, \tau) \subseteq U$ and hence $A$ is $\mathcal{I}$-g-closed in $X$. □

Corollary 2.9 Let $(X, \tau, \mathcal{I})$ be a topological space and $A$ and $F$ subsets of $X$. If $A$ is $\mathcal{I}$-g-closed and $F$ is closed in $(X, \tau)$, then $A \cap F$ is $\mathcal{I}$-g-closed.

Proof. Since $A \cap F$ is closed in $(A, \tau|A)$, then $A \cap F$ is $\mathcal{I}_A$-g-closed in $(A, \tau|A, \mathcal{I}_A)$. By Theorem 2.8, $A \cap F$ is $\mathcal{I}$-g-closed. □

Example 2.10 Corollary 2.9 is not necessarily true if $\gamma$ is an arbitrary operator. Let $(X, \tau)$ be the real line and consider any ideal such that $\tau^*(\mathcal{I}) = \tau^\alpha$. Observe that we have this equality in case when $\mathcal{I} = \mathcal{N}$. Define the following $\gamma$ operation: for any open set $U$, let $\gamma(U) = X$ if the open interval $(0, 1)$ is contained in $U$, otherwise let $\gamma(U) = U$. If $A = (0, 1)$, then $A$ is clearly $(\mathcal{I}, \gamma)$-generalized closed. Now, consider the closed set $B = [\frac{1}{2}, 1]$. Then the intersection of $A$ and $B$ is $[\frac{1}{2}, 1)$, which is contained in the open set $V = (\frac{1}{2}, 1)$. Obviously, the local function of $[\frac{1}{2}, 1)$ with respect to $\mathcal{N}$ is $[\frac{1}{2}, 1]$ and is not contained in $\gamma(V) = V$.

Theorem 2.11 Let $A \subseteq S \subseteq (X, \tau, \mathcal{I}, \gamma)$. If $A \in IG(X)$ and $S \in \tau$, then $A \in IG(S)$.

Proof. Let $U$ be an open subset of $(S, \tau|S)$ such that $A \subseteq U$. Since $S \in \tau$, then $U \in \tau$. Then $A^*(\mathcal{I}) \subseteq U^\gamma$, since $A \in IG(X)$. Using Lemma 2.7, we have $A^*(\mathcal{I}_S, \tau|S) \subseteq U^\gamma|S$, where $U^\gamma|S$ means the image of the operation $\gamma|S: \tau|S \rightarrow \mathcal{P}(S)$, defined by $(\gamma|S)(U) = \gamma(U) \cap S$ for each $U \in \tau|S$. Hence $A \in IG(S)$. □

Theorem 2.12 Let $A$ be a subset of $(X, \tau, \mathcal{I}, \gamma_{id})$. Then, $A$ is $\mathcal{I}$-g-closed if and only if $A^* \setminus A$ does not contain a non-empty closed subset.

Proof. (Necessity) Assume that $F$ is a closed subset of $A^* \setminus A$. Note that clearly $A \subseteq X \setminus F$, where $A$ is $\mathcal{I}$-g-closed and $X \setminus F \in \tau$. Thus $A^* \subseteq X \setminus F$, i.e. $F \subseteq X \setminus A^*$. Since due to our assumption $F \subseteq A^*$, $F \subseteq (X \setminus A^*) \cap A^* = \emptyset$.

(Sufficiency) Let $U$ be an open subset containing $A$. Since $A^*$ is closed [15, Theorem 2.3 (c)] and $A^* \cap (X \setminus U) \subseteq A^* \setminus A$ holds, then $A^* \cap (X \setminus U)$ is a closed set contained in $A^* \setminus A$. By assumption, $A^* \cap (X \setminus U) = \emptyset$ and hence $A^* \subseteq U$. □
Theorem 2.13 Let $A \subseteq (X, \tau, \mathcal{I})$ and $\gamma_1$ and $\gamma_2$ be two operations.

(i) If $A$ is both $(\mathcal{I}, \gamma_1)$-g-closed and $(\mathcal{I}, \gamma_2)$-g-closed, then $A$ is $(\mathcal{I}, \gamma_1 \wedge \gamma_2)$-g-closed, where $\gamma_1 \wedge \gamma_2$ is an operation defined by $(\gamma_1 \wedge \gamma_2)(U) = \gamma_1(U) \cap \gamma_2(U)$ for each $U \in \tau$.

(ii) Under the assumption of (i), if moreover the operators $\gamma_1$ and $\gamma_2$ are mutually dual, then $A$ is $\mathcal{I}$-g-closed.

(iii) Every set $A \subseteq (X, \tau, \mathcal{I})$ is $(\mathcal{I}, \gamma_{cl})$-g-closed. \hfill $\square$

Corollary 2.14 For a set $A \subseteq (X, \tau, \mathcal{I})$, the following conditions are equivalent:

1. $A$ is $(\mathcal{I}, \gamma_f)$-g-closed.
2. $A$ is $\mathcal{I}$-g-closed.

Proof. (1) $\Rightarrow$ (2) By Theorem 2.13 (iii), $A$ is $(\mathcal{I}, \gamma_{cl})$-g-closed. Since $\gamma_f$ and $\gamma_{cl}$ are mutually dual due to [35, Proposition 2], then in the notion of Theorem 2.13, $A$ is $\mathcal{I}$-g-closed.

(2) $\Rightarrow$ (1) is obvious. \hfill $\square$

3 \quad $\gamma$-\(T_{\mathcal{I}}\)-spaces and the digital plane

Definition 3 A space $(X, \tau, \mathcal{I}, \gamma)$ is called an $\gamma$-\(T_{\mathcal{I}}\)-space if every $(\mathcal{I}, \gamma)$-generalized closed subset of $X$ is $\tau^*$-closed. We use the simpler notation $T_{\mathcal{I}}$-space, in case $\gamma$ is the identity operator.

Theorem 3.1 Let $(X, \tau, \mathcal{I}, \gamma)$ be a space and let $A \subseteq X$. Then:

1. $X$ is a $T_{\{\emptyset\}}$-space if and only if $X$ is a $T_{\frac{1}{2}}$-space.
2. $X$ is a $T_N$-space if and only if $X$ is a $T_{\frac{1}{2}}$-space.
3. $X$ is a $\gamma_{ic}$-$T_N$-space if and only if $X$ is discrete.
4. $X$ is a $\gamma_{ic}$-$T_{\{\emptyset\}}$-space if and only if $X$ is discrete.

Proof. (1) follows from Theorem 2.2. (2) follows from Theorem 3.9 from [22] and Theorem 2.2, while (3) follows from Theorem 5.3 from [22] and Theorem 2.2. For (4), note that a space is discrete if and only if every r-g-closed set is closed. \hfill $\square$
Remark 3.2 Note that when $I$ is the maximal ideal $\mathcal{P}(X)$, then every space $(X, \tau, I, \gamma)$ is a $\gamma$-$T_I$-space.

Next we consider the case when $\gamma$ is the identity operator.

Theorem 3.3 For a space $(X, \tau, I)$, the following conditions are equivalent:

(1) $X$ is a $T_I$-space.

(2) Each singleton of $(X, \tau)$ is either closed or $\tau^*(I)$-open.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$. If $\{x\}$ is not closed, then $A = X \setminus \{x\} \notin \tau$ and then $A$ is trivially $I$-g-closed. By (1), $A$ is $\tau^*$-closed. Hence $\{x\}$ is $\tau^*$-open.

(2) $\Rightarrow$ (1) Let $A$ be $I$-g-closed and let $x \in \text{Cl}^*(A)$. We have the following two cases:

Case 1. $\{x\}$ is closed. By Theorem 2.12, $A^* \setminus A$ does not contain a non-empty closed subset. This shows that $x \in A$.

Case 2. $\{x\}$ is $\tau^*$-open. Then $\{x\} \cap A \neq \emptyset$. Hence $x \in A$.

Thus in both cases $x$ is in $A$ and so $A = \text{Cl}^*(A)$, i.e. $A$ is $\tau^*$-closed, which shows that $X$ is a $T_I$-space. □

Corollary 3.4 Every $T_{1/2}$-space is a $T_I$-space. □

Let $(\mathbb{Z}, \kappa)$ be the digital line (= Khalimsky line) [18]. The topology $\kappa$ has $\{\{2n - 1, 2n, 2n + 1\}; n \in \mathbb{Z}\}$ as a subbase. Every singleton is open or closed in $(\mathbb{Z}, \kappa)$. In fact, every singleton $\{2n\}, n \in \mathbb{Z}$ is closed and every singleton $\{2m + 1\}, m \in \mathbb{Z}$ is open. The space $(\mathbb{Z}, \kappa)$ is a typical example of a $T_{1/2}$-space [18] and moreover, it is an example of a $T_{3/4}$-space [5]. Let $(\mathbb{Z}^2, \kappa^2)$ be the digital plane, i.e., the topological product of two digital lines. We define a set $O(\mathbb{Z}^2) = \{(2n + 1, 2m + 1) \in \mathbb{Z}^2; n, m \in \mathbb{Z}\}$. Let $\mathcal{I}(O(\mathbb{Z}^2))$ be the ideal of all subsets of $O(\mathbb{Z}^2)$, cf. [15, Example 2.9].

We will show that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is a $T_{1/2}$-space, where $\mathcal{I}' = \mathcal{I}(O(\mathbb{Z}^2))$.

Theorem 3.5 (i) The space $(\mathbb{Z}^2, \kappa^2, \mathcal{I}')$ is a $T_{1/2}$-space, where $\mathcal{I}' = \mathcal{I}(O(\mathbb{Z}^2))$, and $(\mathbb{Z}^2, \kappa^2)$ is a not $T_{1/2}$.

(ii) The induced space $(\mathbb{Z}^2, (\kappa^2)^*)$ from $(\mathbb{Z}^2, \kappa^2)$ is a $T_{1/2}$-space.
Proof. We will check condition (2) in Theorem 3.3. That is, we will prove that every singleton \( \{x, y\} \) is closed or \((\kappa^2)^*\)-open. For a subset \( A \subseteq \mathbb{Z}^2 \), in the proof below, we denote \( A^*(\mathcal{I}(O(\mathbb{Z}^2)), \kappa^2) \) by \( A^* \).

We consider the following four cases:

Case 1. \((x, y) = (2n + 1, 2m)\): In this case, we claim that the singleton \( \{x, y\} \) is \((\kappa^2)^*\)-open. There exists an open neighborhood \( \{2n + 1\} \times \{2m - 1, 2m, 2m + 1\} \) of \((x, y)\), say \( U \), such that \( U \cap (\mathbb{Z}^2 \setminus \{(x, y)\}) = \{(2n + 1, 2m + 1), (2n + 1, 2m - 1)\} \in \mathcal{I}(O(\mathbb{Z}^2)) \). Then, \((x, y) \notin (\mathbb{Z}^2 \setminus \{(x, y)\})^*\) and so \( (\mathbb{Z}^2 \setminus \{(x, y)\})^* \subseteq (\mathbb{Z}^2 \setminus \{(x, y)\}) \). That is, the singleton \( \{x, y\} \) is \((\kappa^2)^*\)-open in \((\mathbb{Z}^2, \kappa^2)\).

Case 2. \((x, y) = (2n, 2m)\): In this case, we claim that the singleton \( \{x, y\} \) is \((\kappa^2)^*\)-open. There exists an open neighborhood \( \{2n - 1, 2n, 2n + 1\} \times \{2m + 1\} \) of \((x, y)\), say \( U \), such that \( U \cap (\mathbb{Z}^2 \setminus \{(x, y)\}) = \{(2n - 1, 2m + 1), (2n + 1, 2m + 1)\} \in \mathcal{I}(O(\mathbb{Z}^2)) \). Then, \((x, y) \notin (\mathbb{Z}^2 \setminus \{(x, y)\})^*\) and so \( (\mathbb{Z}^2 \setminus \{(x, y)\})^* \subseteq (\mathbb{Z}^2 \setminus \{(x, y)\}) \). That is, the singleton \( \{x, y\} \) is \((\kappa^2)^*\)-open in \((\mathbb{Z}^2, \kappa^2)\).

Case 3. \((x, y) = (2n, 2m)\): The singleton \( \{x, y\} \) is closed and so it is \((\kappa^2)^*\)-closed in \((\mathbb{Z}^2, \kappa^2)\).

Case 4. \((x, y) = (2n + 1, 2m + 1)\): Since \( \{2n + 1, 2m + 1\} \) is open, it is \((\kappa^2)^*\)-open in \((\mathbb{Z}^2, \kappa^2)\).

Therefore, every singleton is closed or \((\kappa^2)^*\)-open. By Theorem 3.3, \((\mathbb{Z}^2, \kappa^2, T')\) is a \(T_{\mathcal{N}}\)-space, where \( T' = \mathcal{I}(O(\mathbb{Z}^2)) \). Clearly, \((\mathbb{Z}^2, \kappa^2)\) is not \( T_{\frac{1}{2}} \).

(ii) By (i), every singleton is open or closed in \((\mathbb{Z}^2, (\kappa^2)^*)\). Therefore, it is \( T_{\frac{1}{2}} \). □

Question. As shown above \( T_{\mathcal{N}}\)-spaces are precisely the \( T_{\frac{1}{2}} \)-spaces. Do the classes of \( T_{\mathcal{M}}, T_{\mathcal{F}^{-}}, T_{C} \) or \( T_{B} \)-spaces coincide with some already known classes of topological spaces (of course weaker than \( T_{\frac{1}{2}} \))?

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