ON sg-CLOSED SETS AND $g\alpha$-CLOSED SETS

Jiling Cao[a], Maximilian Ganster[a,b] and Ivan Reilly[a]

Abstract

In a recent paper, J. Dontchev posed the question of characterizing

(i) the class of spaces in which every semi-preclosed set is sg-closed, and

(ii) the class of spaces in which every preclosed set is $g\alpha$-closed.

In this note, we will show that these classes of spaces coincide and that they consist of precisely those spaces which are the topological sum of a locally indiscrete space and a strongly irresolvable space.

1 Introduction and preliminaries

Recently there has been considerable interest in the study of various forms of generalized closed sets and their relationships to other classes of sets such as α-open sets, semi-open sets and preopen sets. In a recent paper, Dontchev [2] showed that every sg-closed set is semi-preclosed, and that every $g\alpha$-closed set is preclosed. He then posed the problem of characterizing (i) the class of spaces in which every semi-preclosed set is sg-closed, and (ii) the class of spaces in which every preclosed set is $g\alpha$-closed. In this note, we will address this problem by showing that these classes of spaces coincide.

A subset S of a topological space (X, τ) is called *α-open* (*semi-open, preopen, semi-preopen*) if $S \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}S))$ ($S \subseteq \mathrm{cl}(\mathrm{int}S)$, $S \subseteq \mathrm{int}(\mathrm{cl}S)$, $S \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}S))$). Moreover, S is said to be *α-closed* (*semiclosed, preclosed, semi-preclosed*) if $X - S$ is α-open (semi-open, preopen, semi-preopen) or, equivalently, if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}S)) \subseteq S$ ($\mathrm{int}(\mathrm{cl}S) \subseteq S$, $\mathrm{cl}(\mathrm{int}S) \subseteq S$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}S)) \subseteq$

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S). The α-*closure* (*semi-closure, preclosure, semi-preclosure*) of $S \subseteq X$ is the smallest α-closed (semi-closed, preclosed, semi-preclosed) set containing S. It is well known that $\alpha\text{-cl}S = S \bigcup \text{cl}(\text{int}(\text{cl}S))$ and $\text{scl}S = S \bigcup \text{int}(\text{cl}S)$, !!$\text{pcl}S = S \bigcup \text{cl}(\text{int}S)$ and $\text{spcl}S = S \bigcup \text{int}(\text{cl}(\text{int}S))$. The α-*interior* of $S \subseteq X$ is the largest α-open set contained in S, and we have $\alpha\text{-int}S = S \bigcap \text{int}(\text{cl}(\text{int}S))$. It is worth mentioning that the collection of all α-open subsets of (X, τ) is a topology τ^α on X [9] which is finer than τ, and that a subset S is α-open if and only if it is semi-open and preopen [10]. Moreover, (X, τ) and (X, τ^α) share the same class of dense subsets.

Definition 1. A subset A of (X, τ) is called

(1) *generalized closed* (briefly, *g-closed*) [7] if $\text{cl}A \subseteq U$, whenever $A \subseteq U$ and U is open;

(2) *g-open* [7], if $X - A$ is g-closed;

(3) *sg-closed* [1], if $\text{scl}A \subseteq U$ whenever $A \subseteq U$ and U is semi-open;

(4) *sg-open* [1], if $X - A$ is sg-closed;

(5) *gα-closed* [8], if $\alpha\text{-cl}A \subseteq U$ whenever $A \subseteq U$ and U is α-open, or equivalently, if A is g-closed in (X, τ^α).

(6) *gα-open* [8], if $X - A$ is $g\alpha$-closed.

Lemma 1.1. [7] *The union of two g-closed subsets is g-closed. The intersection of a closed subset and a g-closed subset is g-closed.*

Recall that a space (X, τ) is said to be *locally indiscrete* if every open subset is closed. The following observation and its corollary are easily proved.

Lemma 1.2. *Let A be a clopen locally indiscrete subspace of (X, τ). Let $W \subseteq A$ be α-open in (X, τ). Then W is clopen in (X, τ).*

Corollary 1.3. *Let A be a clopen locally indiscrete subspace of (X, τ). Then every subset of A is gα-closed and gα-open in (X, τ).*

Let S be a subset of (X, τ). A *resolution* of S is a pair $< E_1, E_2 >$ of disjoint dense subsets of S. Furthermore, S is said to be *resolvable* if it possesses a resolution, otherwise S is called *irresolvable*. In addition, S is called *strongly irresolvable*, if every open subspace of S is irresolvable. Observe that if $< E_1, E_2 >$ is a resolution of S then E_1 and E_2 are codense in (X, τ), i.e. have empty interior.

Lemma 1.4. [5, 4] *Every space (X, τ) has a unique decomposition $X = F \bigcup G$, where F is closed and resolvable and G is open and hereditarily irresolvable.*

Recall that a space (X, τ) is said to be *submaximal* (*g-submaximal*) if every dense subset is open (g-open). Every submaximal space is g-submaximal, while an indiscrete space is g-submaximal but not submaximal. Note also that every submaximal space is hereditarily irresolvable, and that every dense subspace of cl$G$ is strongly irresolvable, where G is defined in Lemma 1.4.

Lemma 1.5. *Let B be an open, strongly irresolvable subspace of (X, τ), and let $D \subseteq B$ be dense in B. Then $D \in \tau^\alpha$.*

Proof. By Theorem 2 in [4], int$D$ is dense in B, hence $B \subseteq \mathrm{cl}(\mathrm{int}D)$ and so $B \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}D))$. Consequently,

$$D = D \bigcap B \subseteq D \bigcap \mathrm{int}(\mathrm{cl}(\mathrm{int}D)) = \alpha\text{-int}D,$$

i.e. D is α-open in (X, τ). □

Jankovic and Reilly [6] pointed out that every singleton $\{x\}$ of a space (X, τ) is either nowhere dense or preopen. This gives us another decomposition $X = X_1 \bigcup X_2$ of (X, τ), where $X_1 = \{x \in X : \{x\} \text{ is nowhere dense }\}$ and $X_2 = \{x \in X : \{x\} \text{ is preopen }\}$. The usefulness of this decomposition is illustrated by the following result.

Lemma 1.6. [3] *A subset A of (X, τ) is sg-closed if and only if $X_1 \bigcap \mathrm{scl}A \subseteq A$.*

2 Dontchev's questions

We will consider the following two properties of topological spaces:

(P1) Every semi-preclosed set is sg-closed;

(P2) Every preclosed set is $g\alpha$-closed.

We are now able to solve the problem of Dontchev posed in [2], i.e. to characterize the class of spaces satisfying (P1), respectively (P2), in an unexpected way. Note that we will use the decompositions $X = F \bigcup G$ and $X = X_1 \bigcup X_2$ mentioned in Section 1.

Theorem 2.1. *For a space (X, τ) the following are equivalent:*

(1) (X, τ) satisfies (P1),

(2) $X_1 \bigcap \mathrm{scl}A \subseteq \mathrm{spcl}A$ for each $A \subseteq X$,

(3) $X_1 \subseteq \mathrm{int}(\mathrm{cl}G)$,

(4) (X, τ) is the topological sum of a locally indiscrete space and a strongly irresolvable space,

(5) (X, τ) satisfies (P2),

(6) (X, τ^α) is g-submaximal.

Proof. (1) $\Rightarrow$ (2). Let $x \in X_1 \bigcap \mathrm{scl}A$ and suppose that $x \notin \mathrm{spcl}A = B$. Then the semi-preclosed set B is contained in the semi-open set $X - \{x\}$, and therefore $\mathrm{scl}B \subseteq X - \{x\}$. Since $A \subseteq B$ we have $\mathrm{scl}A \subseteq \mathrm{scl}B$, hence $x \notin \mathrm{scl}A$, a contradiction.

(2) $\Rightarrow$ (3). Let $< E_1, E_2 >$ be a resolution of F, and let $D_1 = E_1 \bigcup G$ and $D_2 = E_2 \bigcup G$. Then D_1 and D_2 are dense, $\mathrm{scl}D_1 = \mathrm{scl}D_2 = X$ and $\mathrm{int}D_1 = \mathrm{int}D_2 = G$. Since

$$\mathrm{spcl}D_i = D_i \bigcup \mathrm{int}(\mathrm{cl}G) = E_i \bigcup \mathrm{int}(\mathrm{cl}G) \text{ for } i = 1, 2,$$

by assumption we have

$$X_1 \subseteq (E_1 \bigcup \mathrm{int}(\mathrm{cl}G)) \bigcap (E_2 \bigcup \mathrm{int}(\mathrm{cl}G)) = \mathrm{int}(\mathrm{cl}G).$$

(3) $\Rightarrow$ (4). Let $A = X - \mathrm{int}(\mathrm{cl}G) = \mathrm{cl}(\mathrm{int}F)$, and $B = \mathrm{int}(\mathrm{cl}G)$. Then B is strongly irresolvable and, by assumption, $A \subseteq X_2$. If $C \subseteq A$ is closed in A, then C is closed in X and preopen. Thus C is open in X and hence in A. Therefore A is a clopen locally indiscrete subspace.

(4) $\Rightarrow$ (5). Let $X = A \bigcup B$, where A and B are disjoint and clopen, A is locally indiscrete and B is strongly irresolvable. Let $C \subseteq X$ be preclosed. As a consequence of Proposition 1 in [4], $C = H \bigcup E$, where H is τ-closed, hence τ^α-closed, and E is codense in (X, τ), hence codense in (X, τ^α). Since $(X - E) \bigcap B$ is dense in B, by Lemma 1.5 we have $(X - E) \bigcap B \in \tau^\alpha$. Moreover, $(X - E) \bigcap A$ is g-open in (X, τ^α) by Corollary 1.3. Therefore, by Lemma 1.1, $X - E$ is g-open in (X, τ^α) and $C = H \bigcup E$ is g-closed in (X, τ^α).

(5) $\Rightarrow$ (6). Let $D \subseteq X$ be τ^α-dense. Then $X - D$ is preclosed in (X, τ) and so g-closed in (X, τ^α), i.e. D is g-open in (X, τ^α).

(6) $\Rightarrow$ (3). Let $x \in X_1$ and suppose that $x \notin \mathrm{int}(\mathrm{cl}G)$, i.e. $x \in \mathrm{cl}(\mathrm{int}F)$. Let $< E_1, E_2 >$ be a resolution of $\mathrm{cl}(\mathrm{int}F)$, and without loss of generality let $x \in E_1$. Since E_2 is codense, it is g-closed in (X, τ^α) and contained in the α-open set $X - \{x\}$. Hence

$$\alpha\text{-cl}E_2 = E_2 \bigcup \text{cl}(\text{int}(\text{cl}E_2)) = \text{cl}(\text{int}F) \subseteq X - \{x\},$$

a contradiction.

(3) $\Rightarrow$ (2). Let $x \in X_1 \bigcap \text{scl}A$ and suppose that

$$x \notin \text{spcl}A = A \bigcup \text{int}(\text{cl}(\text{int}A)).$$

Pick an open neighbourhood V of x with $V \subseteq \text{cl}G$ and $V \subseteq \text{cl}A$. Since

$$x \in X - \text{int}(\text{cl}(\text{int}A)) = \text{cl}(\text{int}(\text{cl}(X - A))),$$

we conclude that $H = V \bigcap \text{int}(\text{cl}(X - A))$ is nonempty and open. Now it is easily checked that $< H \bigcap A, H \bigcap (X - A) >$ is a resolution of H, and therefore $H \subseteq \text{int}F$, i.e. $H \bigcap \text{cl}G = \emptyset$, a contradiction to $H \subseteq V \subseteq \text{cl}G$.

(2) $\Rightarrow$ (1). Let A be semi-preclosed, i.e. $A = \text{spcl}A$. By assumption, we have $X_1 \bigcap \text{scl}A \subseteq A$. Hence A is sg-closed by Lemma 1.6. $\square$

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a) Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1, New Zealand

Permanent address of the second author:

b) Department of Mathematics, Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria