ON $sg$-CLOSED SETS AND $ga$-CLOSED SETS

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Abstract

In a recent paper, J. Dontchev posed the question of characterizing
(i) the class of spaces in which every semi-preclosed set is $sg$-closed,
and
(ii) the class of spaces in which every preclosed set is $ga$-closed.
In this note, we will show that these classes of spaces coincide and
that they consist of precisely those spaces which are the topological
sum of a locally indiscrete space and a strongly irresolvable space.

1 Introduction and preliminaries

Recently there has been considerable interest in the study of various forms of
generalized closed sets and their relationships to other classes of sets such as
$\alpha$-open sets, semi-open sets and preopen sets. In a recent paper, Dontchev [2]
showed that every $sg$-closed set is semi-preclosed, and that every $ga$-closed
set is preclosed. He then posed the problem of characterizing (i) the class
of spaces in which every semi-preclosed set is $sg$-closed, and (ii) the class
of spaces in which every preclosed set is $ga$-closed. In this note, we will address
this problem by showing that these classes of spaces coincide.

A subset $S$ of a topological space $(X, \tau)$ is called $\alpha$-open ($semi$-$open$,
preopen, semi-preopen) if $S \subseteq \text{int} (\text{cl} (\text{int} S))$ ($S \subseteq \text{cl} (\text{int} S)$, $S \subseteq \text{int} (\text{cl} S)$,
$S \subseteq \text{cl} (\text{int} (\text{cl} S)))$. Moreover, $S$ is said to be $\alpha$-closed ($semiclosed$, preclosed,
semi-preclosed) if $X - S$ is $\alpha$-open (semi-open, preopen, semi-preopen) or,
equivalently, if $\text{cl} (\text{int} (\text{cl} S)) \subseteq S$ ($\text{int} (\text{cl} S) \subseteq S$, $\text{cl} (\text{int} S) \subseteq S$, $\text{int} (\text{cl} (\text{int} S)) \subseteq$

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The $\alpha$-closure (semi-closure, preclosure, semi-preclosure) of $S \subseteq X$ is the smallest $\alpha$-closed (semi-closed, preclosed, semi-preclosed) set containing $S$. It is well known that $\alpha\text{-cl}S = S \cup \text{cl}(\text{int}(\text{cl}S))$ and $\text{scl}S = S \cup \text{int}(\text{cl}S)$, $\text{!!pcl}S = S \cup \text{cl}(\text{int}S)$ and $\text{spcl}S = S \cup \text{int}(\text{cl}(\text{int}S))$. The $\alpha$-interior of $S \subseteq X$ is the largest $\alpha$-open set contained in $S$, and we have $\alpha\text{-int}S = S \cap \text{int}(\text{cl}(\text{int}S))$.

It is worth mentioning that the collection of all $\alpha$-open subsets of $(X, \tau)$ is a topology $\tau^\alpha$ on $X$ which is finer than $\tau$, and that a subset $S$ is $\alpha$-open if and only if it is semi-open and preopen [10]. Moreover, $(X, \tau)$ and $(X, \tau^\alpha)$ share the same class of dense subsets.

**Definition 1.** A subset $A$ of $(X, \tau)$ is called

1. generalized closed (briefly, $g$-closed) [7] if $\text{cl}A \subseteq U$, whenever $A \subseteq U$ and $U$ is open;
2. $g$-open [7], if $X - A$ is $g$-closed;
3. $sg$-closed [1], if $\text{scl}A \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open;
4. $sg$-open [1], if $X - A$ is $sg$-closed;
5. $\alpha$-closed [8], if $\alpha\text{-cl}A \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open, or equivalently, if $A$ is $g$-closed in $(X, \tau^\alpha)$.
6. $\alpha$-open [8], if $X - A$ is $\alpha$-closed.

**Lemma 1.1.** [7] The union of two $g$-closed subsets is $g$-closed. The intersection of a closed subset and a $g$-closed subset is $g$-closed.

Recall that a space $(X, \tau)$ is said to be locally indiscrete if every open subset is closed. The following observation and its corollary are easily proved.

**Lemma 1.2.** Let $A$ be a clopen locally indiscrete subspace of $(X, \tau)$. Let $W \subseteq A$ be $\alpha$-open in $(X, \tau)$. Then $W$ is clopen in $(X, \tau)$.

**Corollary 1.3.** Let $A$ be a clopen locally indiscrete subspace of $(X, \tau)$. Then every subset of $A$ is $\alpha$-closed and $\alpha$-open in $(X, \tau)$.

Let $S$ be a subset of $(X, \tau)$. A resolution of $S$ is a pair $< E_1, E_2 >$ of disjoint dense subsets of $S$. Furthermore, $S$ is said to be resolvable if it possesses a resolution, otherwise $S$ is called irresolvable. In addition, $S$ is called strongly irresolvable, if every open subspace of $S$ is irresolvable. Observe that if $< E_1, E_2 >$ is a resolution of $S$ then $E_1$ and $E_2$ are codense in $(X, \tau)$, i.e. have empty interior.

**Lemma 1.4.** [5, 4] Every space $(X, \tau)$ has a unique decomposition $X = F \cup G$, where $F$ is closed and resolvable and $G$ is open and hereditarily irresolvable.
Recall that a space \((X, \tau)\) is said to be submaximal \((g\text{-submaximal})\) if every dense subset is open \((g\text{-open})\). Every submaximal space is \(g\text{-submaximal}\), while an indiscrete space is \(g\text{-submaximal} \) but not submaximal. Note also that every submaximal space is hereditarily irresolvable, and that every dense subspace of \(\text{cl}G\) is strongly irresolvable, where \(G\) is defined in Lemma 1.4.

**Lemma 1.5.** Let \(B\) be an open, strongly irresolvable subspace of \((X, \tau)\), and let \(D \subseteq B\) be dense in \(B\). Then \(D \in \tau^\alpha\).

**Proof.** By Theorem 2 in [4], \(\text{int}D\) is dense in \(B\), hence \(B \subseteq \text{cl} (\text{int}D)\) and so \(B \subseteq \text{int} (\text{cl} (\text{int}D))\). Consequently,

\[ D = D \cap B \subseteq D \cap \text{int} (\text{cl} (\text{int}D)) = \alpha \text{-int} D, \]

i.e. \(D\) is \(\alpha\)-open in \((X, \tau)\). \(\Box\)

Jankovic and Reilly [6] pointed out that every singleton \(\{x\}\) of a space \((X, \tau)\) is either nowhere dense or preopen. This gives us another decomposition \(X = X_1 \cup X_2\) of \((X, \tau)\), where \(X_1 = \{x \in X : \{x\}\ \text{is nowhere dense}\}\) and \(X_2 = \{x \in X : \{x\}\ \text{is preopen}\}\). The usefulness of this decomposition is illustrated by the following result.

**Lemma 1.6.** [3] A subset \(A\) of \((X, \tau)\) is \(sg\text{-closed}\) if and only if \(X_1 \cap \text{scl} A \subseteq A\).

## 2  Dontchev’s questions

We will consider the following two properties of topological spaces:

(P1) Every semi-preclosed set is \(sg\text{-closed}\);
(P2) Every preclosed set is \(g\alpha\text{-closed}\).

We are now able to solve the problem of Dontchev posed in [2], i.e. to characterize the class of spaces satisfying (P1), respectively (P2), in an unexpected way. Note that we will use the decompositions \(X = F \cup G\) and \(X = X_1 \cup X_2\) mentioned in Section 1.

**Theorem 2.1.** For a space \((X, \tau)\) the following are equivalent:

1. \((X, \tau)\) satisfies (P1),
2. \(X_1 \cap \text{scl} A \subseteq \text{spcl} A\) for each \(A \subseteq X\),
3. \(X_1 \subseteq \text{int}(\text{cl}G)\),
(4) \((X, \tau)\) is the topological sum of a locally indiscrete space and a strongly irresolvable space,

(5) \((X, \tau)\) satisfies \(\text{(P2)}\),

(6) \((X, \tau^\alpha)\) is \(g\)-submaximal.

Proof. (1) \(\Rightarrow\) (2). Let \(x \in X_1 \cap \text{scl}A\) and suppose that \(x \notin \text{spcl}A = B\). Then the semi-preclosed set \(B\) is contained in the semi-open set \(X - \{x\}\), and therefore \(\text{scl}B \subseteq X - \{x\}\). Since \(A \subseteq B\) we have \(\text{scl}A \subseteq \text{scl}B\), hence \(x \notin \text{scl}A\), a contradiction.

(2) \(\Rightarrow\) (3). Let \(< E_1, E_2 >\) be a resolution of \(F\), and let \(D_1 = E_1 \cup G\) and \(D_2 = E_2 \cup G\). Then \(D_1\) and \(D_2\) are dense, \(\text{scl}D_1 = \text{scl}D_2 = X\) and \(\text{int}D_1 = \text{int}D_2 = G\). Since

\[
\text{spcl}D_i = D_i \cup \text{int}(\text{cl}G) = E_i \cup \text{int}(\text{cl}G) \quad \text{for} \quad i = 1, 2,
\]

by assumption we have

\[
X_1 \subseteq (E_1 \cup \text{int}(\text{cl}G)) \cap (E_2 \cup \text{int}(\text{cl}G)) = \text{int}(\text{cl}G).
\]

(3) \(\Rightarrow\) (4). Let \(A = X - \text{int}(\text{cl}G) = \text{cl}(\text{int}F)\), and \(B = \text{int}(\text{cl}G)\). Then \(B\) is strongly irresolvable and, by assumption, \(A \subseteq X_2\). If \(C \subseteq A\) is closed in \(A\), then \(C\) is closed in \(X\) and preopen. Thus \(C\) is open in \(X\) and hence in \(A\). Therefore \(A\) is a clopen locally indiscrete subspace.

(4) \(\Rightarrow\) (5). Let \(X = A \cup B\), where \(A\) and \(B\) are disjoint and clopen, \(A\) is locally indiscrete and \(B\) is strongly irresolvable. Let \(C \subseteq X\) be preclosed. As a consequence of Proposition 1 in [4], \(C = H \cup E\), where \(H\) is \(\tau\)-closed, hence \(\tau^\alpha\)-closed, and \(E\) is codense in \((X, \tau)\), hence codense in \((X, \tau^\alpha)\). Since \((X - E) \cap B\) is dense in \(B\), by Lemma 1.5 we have \((X - E) \cap B \in \tau^\alpha\). Moreover, \((X - E) \cap A\) is \(g\)-open in \((X, \tau^\alpha)\) by Corollary 1.3. Therefore, by Lemma 1.1, \(X - E\) is \(g\)-open in \((X, \tau^\alpha)\) and \(C = H \cup E\) is \(g\)-closed in \((X, \tau^\alpha)\).

(5) \(\Rightarrow\) (6). Let \(D \subseteq X\) be \(\tau^\alpha\)-dense. Then \(X - D\) is preclosed in \((X, \tau)\) and so \(g\)-closed in \((X, \tau^\alpha)\), i.e. \(D\) is \(g\)-open in \((X, \tau^\alpha)\).

(6) \(\Rightarrow\) (3). Let \(x \in X_1\) and suppose that \(x \notin \text{int}(\text{cl}G)\), i.e. \(x \in \text{cl}(\text{int}F)\). Let \(< E_1, E_2 >\) be a resolution of \(\text{cl}(\text{int}F)\), and without loss of generality let \(x \in E_1\). Since \(E_2\) is codense, it is \(g\)-closed in \((X, \tau^\alpha)\) and contained in the \(\alpha\)-open set \(X - \{x\}\). Hence
\[ \alpha \text{-cl} E_2 = E_2 \bigcup \text{cl}(\text{int}(\text{cl} E_2)) = \text{cl}(\text{int} F) \subseteq X - \{x\}, \]
a contradiction.

(3) \Rightarrow (2). Let \( x \in X_1 \cap \text{scl} A \) and suppose that
\[ x \not\in \text{spcl} = A \bigcup \text{int}(\text{cl}(A)). \]
Pick an open neighbourhood \( V \) of \( x \) with \( V \subseteq \text{cl} G \) and \( V \subseteq \text{cl} A \). Since
\[ x \in X - \text{int}(\text{cl}(\text{int} A)) = \text{cl}(\text{int}(X - A)), \]
we conclude that \( H = V \cap \text{int}(\text{cl}(X - A)) \) is nonempty and open. Now it is easily checked that \(< H \cap A, H \cap (X - A) >\) is a resolution of \( H \), and therefore \( H \subseteq \text{int} F \), i.e. \( H \cap \text{cl} G = \emptyset \), a contradiction to \( H \subseteq V \subseteq \text{cl} G \).

(2) \Rightarrow (1). Let \( A \) be semi-preclosed, i.e. \( A = \text{spcl} A \). By assumption, we have \( X_1 \cap \text{scl} A \subseteq A \). Hence \( A \) is \( \text{sg}- \)closed by Lemma 1.6.

\[ \square \]

References


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