Submaximality, Extremal Disconnectedness and Generalized Closed Sets

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Abstract

In this paper, we continue the study of generalized closed sets in a topological space. In particular, we study the question when some classes of generalized closed sets coincide. A new class of spaces, the class of sg-submaximal spaces, is also introduced. Characterizations of extremally disconnected spaces and sg-submaximal spaces are established via various kinds of generalized closed sets.

1 Introduction

During the last few years the study of generalized closed sets has found considerable interest among general topologists. One reason is these objects are natural generalizations of closed sets. More importantly, generalized closed sets suggest some new separation axioms which have been found to be very useful in the study of certain objects of digital topology, for example, the digital line (see e.g. [8]). In [5], Dontchev summarized in a diagram the fundamental relationships between the various types of generalized closed sets. Concerning this diagram, he also posed two questions which have been answered by Cao, Ganster and Reilly in [4]. In the present paper, we take another look at Dontchev's diagram. In doing so, we are able to characterize extremally disconnected spaces via generalized closed sets. Moreover, we introduce the notion of sg-submaximal spaces and investigate this class of

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spaces in terms of the Hewitt decomposition of a topological space. This enables us to provide an example of an sg-submaximal space which! ! is not g-submaximal.

A subset S of a topological space (X, τ) is called *α-open* [resp. *semi-open, preopen, semi-preopen*] if $S \subseteq \text{int}(\text{cl}(\text{int}S))$ [resp. $S \subseteq \text{cl}(\text{int}S)$, $S \subseteq \text{int}(\text{cl}S)$, $S \subseteq \text{cl}(\text{int}(\text{cl}S))$]. Moreover, S is said to be *α-closed* [resp. *semi-closed, preclosed, semi-preclosed*] if $X \setminus S$ is α-open [resp. semi-open, preopen, semi-preopen] or, equivalently, if $\text{cl}(\text{int}(\text{cl}S)) \subseteq S$ [resp. $\text{int}(\text{cl}S) \subseteq S$, $\text{cl}(\text{int}S) \subseteq S$, $\text{int}(\text{cl}(\text{int}S)) \subseteq S$]. The *$\alpha$-closure* [resp. *semi-closure, pre-closure, semi-preclosure*] of $S \subseteq X$ is the smallest α-closed [resp. semi-closed, preclosed, semi-preclosed] set containing S. It is well-known that $\alpha\text{-cl}S = S \bigcup \text{cl}(\text{int}(\text{cl}S))$ and! ! $\text{scl}S = S \bigcup \text{int}(\text{cl}S)$, $\text{pcl}S = S \bigcup \text{cl}(\text{int}S)$ and $\text{spcl}S = S \bigcup \text{int}(\text{cl}(\text{int}S))$. The *$\alpha$-interior* of $S \subseteq X$ is the largest α-open set contained in S, and we have $\alpha\text{-int}S = S \bigcap \text{int}(\text{cl}(\text{int}S))$. It is worth mentioning that the collection of all α-open subsets of (X, τ) is a topology τ^α on X [15] which is finer than τ, and that a subset S is α-open if and only if it is semi-open and preopen [16]. Moreover, (X, τ) and (X, τ^α) share the same class of dense subsets.

Definition 1. A subset A of (X, τ) is called

(1) *generalized closed* (briefly, *g-closed*) [12] if $\text{cl}A \subseteq U$, whenever $A \subseteq U$ and U is open;

(2) *g-open* [12], if $X \setminus A$ is g-closed;

(3) *sg-closed* [3], if $\text{scl}A \subseteq U$ whenever $A \subseteq U$ and U is semi-open;

(4) *sg-open* [3], if $X \setminus A$ is sg-closed;

(5) *$g\alpha$-closed* [13], if $\alpha\text{-cl}A \subseteq U$ whenever $A \subseteq U$ and U is α-open, or equivalently, if A is g-closed in (X, τ^α);

(6) *gs-closed* [2] if $\text{scl}A \subseteq U$ whenever $A \subseteq U$ and U is open.

In [5], a diagram has been provided in order to illustrate the relationships between these classes of generalized closed sets. Let S be a subset of (X, τ). A *resolution* of S is a pair $< E_1, E_2 >$ of disjoint dense subsets of S. Furthermore, S is said to be *resolvable* if it possesses a resolution, otherwise S is called *irresolvable*. In addition, S is called *strongly irresolvable*, if every open subspace of S is irresolvable. Observe that if $< E_1, E_2 >$ is a resolution of S then E_1 and E_2 are codense in (X, τ), i.e. have empty interior.

Lemma 1.1. *[10, 9] Every space (X, τ) has a unique decomposition $X = F \bigcup G$, where F is closed and resolvable and G is open and hereditarily irresolvable.*

Recall that a space (X, τ) is said to be *submaximal* [*g-submaximal*] if every dense subset is open [g-open]. Obviously, every submaximal space is g-submaximal. Note that every submaximal space is hereditarily irresolvable, and that every dense subspace of $\mathrm{cl}G$ is strongly irresolvable, where G is defined in Lemma 1.1. Any indiscrete space with at least two points is g-submaximal but not submaximal.

In [11], Janković and Reilly pointed out that every singleton $\{x\}$ of a space (X, τ) is either nowhere dense or preopen. This provides another decomposition $X = X_1 \bigcup X_2$ of (X, τ), where $X_1 = \{x \in X : \{x\}$ is nowhere dense $\}$ and $X_2 = \{x \in X : \{x\}$ is preopen $\}$. The usefulness of this decomposition is illustrated by the following result which will be used extensively in the sequel.

Theorem 1.2. *Let (X, τ) be a space, and A be a subset of X. Then*

(1) [6] A is sg-closed if and only if $X_1 \bigcap \mathrm{scl}A \subseteq A$.

(2) $\mathrm{pcl}A \subseteq X_1 \bigcup A$.

Proof. (1) has been proved in [6]. Now let $x \in \mathrm{spcl}A$ and suppose that $x \notin X_1$. Then $\{x\}$ is preopen and thus $\{x\} \bigcap A \neq \emptyset$, i.e., $x \in A$. This proves (2). □

Throughout this paper, $X = F \bigcup G$ and $X = X_1 \bigcup X_2$ will always denote the Hewitt decomposition of (X, τ) and the decomposition due to Jankovic and Reilly, respectively.

It is clear that every sg-closed subset of a space (X, τ) is gs-closed. However the converse is not true in general. In [14], Maki at al called spaces whose gs-closed sets are sg-closed T_{gs}-spaces. The following result characterizes the class of T_{gs}-spaces.

Theorem 1.3. *The following are equivalent for a space (X, τ):*

(1) (X, τ) is a T_{gs}-space,

(2) every singleton $\{x\}$ of X is either closed or preopen.

Proof. (1) $\Rightarrow$ (2). Let $x \in X_1$ and suppose that $\{x\}$ is not closed. Then $X \setminus \{x\}$ is gs-closed, dense and semi-open. Hence $X_1 \bigcap \mathrm{scl}(X \setminus \{x\}) = X_1 \subseteq X \setminus \{x\}$, a contradiction.

(2) $\Rightarrow$ (1). Let A be gs-closed and let $x \in X_1 \bigcap \mathrm{scl}A$. Then $\{x\}$ is closed. If $x \notin A$, i.e. $A \subseteq X \setminus \{x\}$. Then $\mathrm{scl}A \subseteq X \setminus \{x\}$, a contradiction. □

2 Some characterizations of extremally disconnected spaces

Recall that a space (X, τ) is *extremally disconnected* if the closure of every open subset of X is open. We first note the following result which is essentially due to Njåstad [15].

Lemma 2.1. *The following are equivalent for a space* (X, τ) *:*
(1) (X, τ) *is extremally disconnected,*
(2) $\mathrm{scl}(A \bigcup B) = \mathrm{scl}A \bigcup \mathrm{scl}B$ *for all* $A, B \subseteq X$*.*
(3) the union of two semi-closed sets is semi-closed.

As an immediate consequence we now have the following theorem.

Theorem 2.2. *The following are equivalent for a space* (X, τ) *:*
(1) (X, τ) *is extremally disconnected,*
(2) the union of two sg*-closed sets is* sg*-closed.*

Proof. (1) $\Rightarrow$ (2). Let A and B be sg-closed. Then, by Theorem 1.2 and Lemma 2.1,

$$\begin{aligned} X_1 \textstyle\bigcap \mathrm{scl}(A \bigcup B) &= X_1 \textstyle\bigcap (\mathrm{scl}A \bigcup \mathrm{scl}B) \\ &= (X_1 \textstyle\bigcap \mathrm{scl}A) \bigcup (X_1 \bigcap \mathrm{scl}B) \subseteq A \bigcup B. \end{aligned}$$

Therefore $A \bigcup B$ is sg-closed, by Theorem 1.2.

(2) $\Rightarrow$ (1). Let U be open and suppose there exists $x \in X$ with $x \in \mathrm{cl}U \setminus \mathrm{int}(\mathrm{cl}U)$. If $S_1 = U \bigcup \{x\}$ and $S_2 = (X \setminus \mathrm{cl}U) \bigcup \{x\}$, then S_1 and S_2 are semi-open, hence sg-open. By assumption, $S_1 \bigcap S_2 = \{x\}$ is sg-open, i.e. $D = X \setminus \{x\}$ is sg-closed. Clearly $x \in X_1$ and so D is dense, i.e. $\mathrm{scl}D = X$. Thus $X_1 \bigcap \mathrm{scl}D = X_1 \subseteq X \setminus \{x\}$, a contradiction. $\square$

Refering to the diagram in [5] we now consider possible converses of some implications in that diagram thereby obtaining some more characterizations of extremally disconnected spaces by using some classes of generalized closed sets.

Theorem 2.3. *For a space* (X, τ) *the following are equivalent:*
(1) (X, τ) *is extremally disconnected,*
(2) every semi-preclosed subset of X *is preclosed,*
(3) every sg*-closed subset of* X *is preclosed,*

(4) every semi-closed subset of X is preclosed,
(5) every semi-closed subset of X is α-closed,
(6) every semi-closed subset of X is $g\alpha$-closed.

Proof. Both (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) are obvious.

(1) $\Rightarrow$ (2). If S is semi-preclosed, then $S = \mathrm{spcl}S = S \bigcup \mathrm{int}(\mathrm{cl}(\mathrm{int}S))$. Since X is extremally disconnected, then $\mathrm{cl}(\mathrm{int}S) = \mathrm{int}(\mathrm{cl}(\mathrm{int}S))$. Then $S = \mathrm{pcl}S$, i.e. S is preclosed.

(2) $\Rightarrow$ (3). From Theorem 2.4 of [5], every sg-closed subset of X is semi-preclosed.

(4) $\Rightarrow$ (5). This follows directly from the result in [16] that a subset is α-closed if and only if it is both semi-closed and preclosed.

(6) $\Rightarrow$ (1). It suffices to show that every regular open set S is closed. To this end, let S be regular open. Then S is both semi-closed and α-open. By (6), S is $g\alpha$-closed which implies that α-$\mathrm{cl}S \subseteq S$. On the other hand α-$\mathrm{cl}S = S \bigcup \mathrm{cl}(\mathrm{int}(\mathrm{cl}S))$. It follows that $S = \mathrm{cl}(\mathrm{int}(\mathrm{cl}S))$. Therefore, S is closed, and (X, τ) is extremally disconnected. □

It has been shown in [4] that (X, τ^α) is g-submaximal if and only if every preclosed subset is $g\alpha$-closed. As a variation of that result we now obtain

Theorem 2.4. *For a space (X, τ) the following are equivalent:*
(1) every semi-preclosed set is $g\alpha$-closed,
(2) (X, τ^α) is extremally disconnected and (X, τ^α) is g-submaximal,

Proof. (1) $\Rightarrow$ (2). If every semi-preclosed set is $g\alpha$-closed, then every semi-closed set is $g\alpha$-closed and hence by Theorem 2.3, (X, τ) is extremally disconnected, so is (X, τ^α). Let $A \subseteq X$ be τ^α-dense. Then $X \setminus A$ is semi-preclosed in (X, τ), and therefore $g\alpha$-closed. Thus A is g-open in (X, τ^α).

(2) $\Rightarrow$ (1). Let A be a semi-preclosed subset of (X, τ). Since (X, τ) is extremally disconnected, then by Theorem 2.3, A is preclosed. It follows from Theorem 2.2 in [4] that A is $g\alpha$-closed. □

3 sg-Submaximal spaces

Observe that we show in Theorem 2.2 that a space is extremally disconnected if and only if the union of two sg-closed sets is sg-closed. In general, however, the union of two sg-closed sets fails to be sg-closed [3]. On the other hand, we do have

Lemma 3.1. *[7] The union of a closed subset and an sg-closed subset is sg-closed.*

Let us call a space (X, τ) *sg-submaximal* [α-submaximal] if every dense subset is sg-open [α-open]. We note that, since (X, τ) and (X, τ^α) share their classes of dense subsets, (X, τ) is α-submaximal if and only if (X, τ^α) is submaximal. This property has been characterized by Ganster [9, Theorem 4]. In the following, we will characterize sg-submaximality in terms of generalized closed sets. We start with a lemma about the Hewitt decomposition.

Lemma 3.2. *Let $E \subseteq X$ be a codense subset of a space (X, τ), i.e.* $\mathrm{int}E = \emptyset$. *Then* $\mathrm{int}(\mathrm{cl}E) \bigcap \mathrm{cl}G = \emptyset$.

Proof. Since $X \setminus E$ is dense in X, $(X \setminus E) \bigcap G$ is dense in G. By Theorem 2 in [9], $\mathrm{int}(X \setminus E) \bigcap G = (X \setminus \mathrm{cl}E) \bigcap G$ is dense in G, i.e. $G \subseteq \mathrm{cl}((X \setminus \mathrm{cl}E) \bigcap G)$. Then,

$$\begin{aligned} \mathrm{int}(\mathrm{cl}E) \bigcap G &\subseteq \mathrm{cl}((X \setminus \mathrm{cl}E) \bigcap G) \bigcap \mathrm{int}(\mathrm{cl}E) \\ &\subseteq \mathrm{cl}((X \setminus \mathrm{cl}E) \bigcap G \bigcap \mathrm{int}(\mathrm{cl}E)) = \emptyset. \end{aligned}$$

and so $\mathrm{int}(\mathrm{cl}E) \bigcap \mathrm{cl}G = \emptyset$. □

Theorem 3.3. *For a space (X, τ) the following are equivalent:*

(1) $X_1 \subseteq \mathrm{cl}G$,

(2) every preclosed subset is sg-closed,

(3) (X, τ) *is sg-submaximal,*

(4) (X, τ^α) *is sg-submaximal,*

Proof. (1) ⇒ (2). Let $C \subseteq X$ be preclosed. By Proposition 1 in [9], $C = H \bigcup E$, where H is closed and E is codense in (X, τ). We have that

$$X_1 \bigcap \mathrm{scl}E = X_1 \bigcap (E \bigcup \mathrm{int}(\mathrm{cl}E)) = (X_1 \bigcap E) \bigcup (X_1 \bigcap \mathrm{int}(\mathrm{cl}E)).$$

By assumption and Lemma 3.2, $X_1 \bigcap \mathrm{int}(\mathrm{cl}E) = \emptyset$. Hence, $X_1 \bigcap \mathrm{scl}E \subseteq E$ which implies that E is sg-closed. By Lemma 3.1, C is sg-closed.

(2) ⇒ (3). Let D be a τ-dense subset of X. Then $X \setminus D$ is preclosed in (X, τ), and so is sg-closed. Therefore, D is sg-open and (X, τ) is sg-submaximal.

(3) ⇔ (4) is obvious, since τ and τ^α share the classes of dense subsets and the classes of sg-open subsets (see [1]).

(4) $\Rightarrow$ (1). Let $x \in X_1$ and suppose that $x \notin \text{cl}G$, i.e. $x \in \text{int}F$. Let $< E_1, E_2 >$ be a resolution of int$F$, and without loss of generality let $x \in E_1$. Since E_2 is codense, it is sg-closed in (X, τ^α) and contained in the α-open set $X \backslash \{x\}$. Hence scl$E_2 = E_2 \bigcup \text{int}(\text{cl}E_2) = \text{int}F \subseteq X \backslash \{x\}$, a contradiction. □

It was pointed out in Theorem 2.1 in [4] that (X, τ^α) is g-submaximal if and only if $X_1 \subseteq \text{int}(\text{cl}G)$. We therefore have

Corollary 3.4. *If (X, τ^α) is g-submaximal, then (X, τ^α) is sg-submaximal.*

Our final example shows that the converse of Corollary 3.4 is not true in general.

Example 3.5. There is an sg-submaximal space which is not g-submaximal. It is well-known that there exists a Hausdorff, dense-in-self, submaximal topology σ on $\mathbf{R}$, the set of reals. Let Y be an infinite set disjoint from $\mathbf{R}$ and let $X = \mathbf{R} \bigcup Y$. We now define a topology τ on X. If $x \in \mathbf{R} \backslash \{0\}$, a basic neighbourhood of x is a σ-open set containing x but not 0. If $x = 0$, a basic neighbourhood of x has the form $U \bigcup Y$, where U is a σ-open set containing x. If $y \in Y$ then y has Y as a minimal open neighbourhood. Now it is easily checked that $X_1 = \mathbf{R}$, $X_2 = Y$, $F = Y \bigcup \{0\}$ and $G = \mathbf{R} \setminus \{0\}$. Moreover, cl$G = \mathbf{R}$ and int(cl$G$) $= G$. Therefore, $X_1 \subseteq \text{cl}G$ but $X_1 \nsubseteq \text{int}(\text{cl}G)$, and so (X, τ^α) is sg-submaximal but not g-submaximal.

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